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心理物理學相似律所含參數之函數性質於費區納表徵下之探究

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中 文 摘 要 ： 此計畫屬心理物理學之理論研究，我們探討在費區納表徵下，作用於韋伯敏感度的「相似律」屬性。有別於之前做法，我們從數學「規律變異」角度切入，探討此「相似律」所含參數的函數特徵。我們並將本研究結果與 Iverson (2006) 研究結果作進一步對照。

中 文 關 鍵 詞 ： 費區納表徵, 函數方程, 相似律, 韋伯敏感度, 規律變異

英 文 摘 要 ： This project concerns a theoretical study of psychophysics. We investigate the properties of the law of similarity (on the Weber sensitivities) within the Fechnerian framework. Instead of assuming differentiability, we rely on regular variation to characterize the parameters in the law of similarity. We also link our results to those in Iverson (2006).

英 文 關 鍵 詞 ： Fechnerian representation, functional equation, law of similarity, Weber sensitivity, regular variation

Final Report

1 Introduction

Let $\xi_s(x) = x + \Delta_s(x)$, where s represents a discrimination index and Δ_s is the 'just noticeable difference' of the stimulus x, be the (Weber) sensitivities in psychophysics. Iverson (2006) introduced a *law of similarity* on ξ_s and studied its impact on the Fechnerian representation. This law states that

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x), \tag{1}$$

in which $\lambda > 0$, $\gamma(\lambda, s)$ and $\eta(\lambda, s)$ are continuous in the two variables, $\gamma(\lambda, s)$ varies monotonically with λ, and $\eta(\lambda, s)$ varies monotonically with s. Further, the following boundary conditions are imposed: $\gamma(1, s) = 1$, $\eta(1, s) = s$, $\gamma(\lambda, 0) = \lambda$, and $\eta(\lambda, 0) = 0$. A special form of (1) assuming that $\gamma(\lambda, s) = \lambda^{\iota(s)}$ was investigated by Hsu and Iverson (2016) to show its impact on the (weakly balanced) affine representation $\Psi(x, y) = F\left(\frac{u(x)-u(y)}{\sigma(y)}\right)$, in which F, u are strictly increasing and σ is positive, and all are continuous and differentiable (see also Hsu, Iverson, & Doble, 2010).

Equation (1) is an interesting and yet complicated functional equation. Working jointly with Dr. Chris Doble from the US, in this report we summarize our attempt at characterizing (1) from a different perspective. Specifically, differentiability regarding η and γ in (1) is not assumed *a priori* here. Instead, we rely on the property of *regular variation* for most of the derivations.

2 Properties of the law of similarity

In Hsu and Iverson (2016), η is postulated to satisfy the multiplicative translation axiom: $\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s))$ (Aczél, 1966), which implies that η is permutable in λ. Indeed, $\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s))$ and $\eta(\lambda'\lambda, s) = \eta(\lambda, \eta(\lambda', s))$ so that $\eta(\lambda', \eta(\lambda, s)) = \eta(\lambda, \eta(\lambda', s))$.

As shown in Hsu and Iverson (2016), η and γ in (1) are somewhat related. To see this, we iterate (1), yielding

$$\xi_s(\lambda\lambda' x) = \gamma(\lambda\lambda', s)\xi_{\eta(\lambda\lambda',s)}(x) \tag{2}$$

$$= \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(\lambda' x) = \gamma(\lambda, s)\gamma(\lambda', \eta(\lambda, s))\xi_{\eta(\lambda',\eta(\lambda,s))}(x). \tag{3}$$

If $\eta(\lambda, s)$ in (1) does not vary with λ, then the boundary condition $\eta(1, s) = s$ implies that $\eta(\lambda, s) = s$ for all λ. It then follows from (2) and (3) that $\gamma(\lambda\lambda', s) = \gamma(\lambda, s)\gamma(\lambda', s)$ and so $\gamma(\lambda, s) = \lambda^{\iota(s)}$ for some function $\iota(s)$. If $\eta(\lambda, s)$ varies with λ for all $s \neq 0$, then from (2), (3) and the postulate that η is multiplicatively translational, we have that there exist continuous and strictly monotonic functions H satisfying $H(0) = 0$ such that $\eta(\lambda, s) = H(\lambda H^{-1}(s))$.[1] Moreover, fix $s = s_0 \neq 0$ and denote $K_{s_0}(\lambda) = \gamma(\lambda, s_0)$ and $J_{s_0}(\lambda) = \eta(\lambda, s_0) = H(\lambda H^{-1}(s_0))$. This leads to $\gamma(\lambda', J_{s_0}(\lambda)) = \frac{K_{s_0}(\lambda\lambda')}{K_{s_0}(\lambda)}$. One can easily check that in both cases η is permutable.

More can be derived about γ in (1). Indeed, applying commutativity of ξ to (1) yields

$$\xi_s(\xi_t(\lambda x)) = \xi_t(\xi_s(\lambda x)), \tag{4}$$

$$\xi_s(\gamma(\lambda, t)\xi_{\eta(\lambda,t)}(x)) = \xi_t(\gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)), \tag{5}$$

$$\gamma(\gamma(\lambda, t), s)\xi_{\eta(\gamma(\lambda,t),s)}(\xi_{\eta(\lambda,t)}(x)) = \gamma(\gamma(\lambda, s), t)\xi_{\eta(\gamma(\lambda,s),t)}(\xi_{\eta(\lambda,s)}(x)). \tag{6}$$

To facilitate the development, we now postulate that γ is permutable in s, i.e., $\gamma(\gamma(\lambda, s), s') = \gamma(\gamma(\lambda, s'), s)$. Then (6) implies that

$$\xi_{\eta(\gamma(\lambda,t),s)}(\xi_{\eta(\lambda,t)}(x)) = \xi_{\eta(\gamma(\lambda,s),t)}(\xi_{\eta(\lambda,s)}(x)). \tag{7}$$

We note that Equation (7) holds if $\eta(\gamma(\lambda, t), s) = \eta(\lambda, s)$, which implies either (i) η does not depend on its first variable, in which case $\eta(\lambda, s) = s$ for all λ and $\gamma(\lambda, s) = \lambda^{\iota(s)}$ for some function $\iota(s)$, or (ii) η depends on its first variable, in which case $\gamma(\lambda, s) = \lambda$ for all s. These results are consistent with Theorem 5 of Hsu and Iverson (2016).

Equation (7) also holds in other peculiar ways without assuming $\eta(\gamma(\lambda, t), s) = \eta(\lambda, s)$. To get an intuition, consider the particular form $\gamma(\lambda, s) = [e^{s/\alpha}(\lambda^\beta - 1) + 1]^{1/\beta}$ and $\eta(\lambda, s) = -\alpha \ln[\lambda^{-\beta}(e^{-s/\alpha} - 1) + 1]$ discussed in the last paragraph of Section 6 of Hsu and Iverson (2016).[2] A simple computation reveals that γ (and η, respectively) satisfies permutability in s (and in λ, respectively). Since both were derived within the Fechnerian framework, (7) must hold with these two forms. To see this, we first note from Falmagne (1985) that being Fechnerian implies that

$$\xi_s(\xi_t(x)) = \xi_{s+t}(x). \tag{8}$$

Applying (8) to (7) and inserting the forms of γ and η then yields

$$\eta(\gamma(\lambda, t), s) + \eta(\lambda, t) = -\alpha \ln[\lambda^{-\beta}(e^{\frac{-(s+t)}{\alpha}} - 1) + 1] \tag{9}$$

$$= \eta(\gamma(\lambda, s), t) + \eta(\lambda, s) = \eta(\lambda, s + t). \tag{10}$$

[1]Note that in the last statement of Theorem 1 in Hsu and Iverson (2016), the expression of H and γ holds only for $s \neq 0$.

[2]This was done by first setting $T = 1$ in (15a) of Iverson (2006): $\xi_s(x) = T(e^{s/\alpha}(x/T)^\beta + (e^{s/\alpha} - 1))^{1/\beta}$. We applied the form to $\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)$ and found that on both sides there is only one term involving x^β. We thus have two equations and two unknown (γ and η) to solve for.

We also recall that η is assumed to be multiplicatively translational. Thus for the case in which $\eta(\lambda, s)$ is non-constant in λ, we apply $\eta(\lambda, s) = H(\lambda H^{-1}(s))$ to (9) and (10) and obtain

$$H(\gamma(\lambda, s)H^{-1}(t)) + H(\lambda H^{-1}(s)) = H(\lambda H^{-1}(s + t)). \tag{11}$$

The form of γ and η described above satisfies (11) — we obtain $H(s) = -\alpha \ln(ks^{-\beta} + 1)$ for some constant $k \neq 0$.

Following (9) and (10), we now focus on solving

$$H(\gamma(\lambda, t)H^{-1}(s)) + H(\lambda H^{-1}(t)) = H(\gamma(\lambda, s)H^{-1}(t)) + H(\lambda H^{-1}(s)). \tag{12}$$

We know that if $\gamma(\lambda, s)$ is a function only of λ, that is, $\gamma(\lambda, s) = h(\lambda)$ for some strictly monotonic function h, then $\gamma(\lambda, s) = \lambda$ for all s. (We know this because of the boundary condition $\gamma(\lambda, 0) = \lambda$.) In such a case, we get Equations (13a)-(13c) of Iverson (2006).

So, suppose that $\gamma(\lambda, s)$ varies with both λ and s. Suppose also that γ is monotonic in each variable, except for $\lambda = 1$, in which case $\gamma(1, s) = 1$ for all s. Letting $u = H^{-1}(s)$ and $v = H^{-1}(t)$, we rewrite (12) as

$$H[\gamma(\lambda, H(v))u] + H(\lambda v) = H[\gamma(\lambda, H(u))v] + H(\lambda u)\,. \tag{13}$$

Theorem 1 in Hsu and Iverson (2016) states that $\gamma(\lambda, s) = \frac{W(\eta(\lambda,s))}{W(s)}$ for some strictly monotonic function W. Thus

$$\gamma(\lambda, H(q)) = \frac{W(\eta(\lambda, H(q)))}{W(H(q))} = \frac{W(H(\lambda H^{-1}(H(q))))}{W(H(q))} = \frac{W(H(\lambda q))}{W(H(q))}, \tag{14}$$

and, with $T = W \circ H$, (13) can be re-expressed as

$$H\left(\frac{T(\lambda v)}{T(v)}u\right) + H(\lambda v) = H\left(\frac{T(\lambda u)}{T(u)}v\right) + H(\lambda u)\,. \tag{15}$$

The following table shows the solutions of $\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)$ obtained in Iverson (2006b). It also shows the respective H obtained from $\eta(\lambda, s) = H(\lambda H^{-1}(s))$ and the T obtained from $\gamma(\lambda, H(q)) = \frac{T(\lambda q)}{T(q)}$, where it is also assumed that $T(1) = 1$.

$\xi_s(x)$	$\gamma(\lambda, s)$	$\eta(\lambda, s)$	$H(s)$	$T(\lambda)$
11a, 11b, 11c				
$e^{s/\alpha}x$	λ	s	–	–
$x^{e^{s/\alpha}}$	$\lambda^{e^{s/\alpha}}$	s	–	–
$x^{e^{-s/\alpha}}$	$\lambda^{e^{-s/\alpha}}$	s	–	–
13a, 13b, 13c				
$e^{s/\alpha}x$	λ	s	–	–
$(x^\beta + \frac{s}{\alpha})^{\frac{1}{\beta}}$	λ	$\lambda^{-\beta}s$	$ks^{-\beta}$	λ
$\left(\frac{1}{x^{-\beta}-\frac{s}{\alpha}}\right)^{\frac{1}{\beta}}$	λ	$\lambda^\beta s$	ks^β	λ
15a, 15b, 15c, 15d, 15e, 15f				
$(e^{s/\alpha}x^\beta + e^{s/\alpha} - 1)^{\frac{1}{\beta}}$	$(e^{s/\alpha}(\lambda^\beta - 1) + 1)^{\frac{1}{\beta}}$	$-\alpha\log(\lambda^{-\beta}(e^{-s/\alpha} - 1) + 1)$	$-\alpha\log(ks^{-\beta} + 1)$	$\left(\frac{\lambda^\beta + k}{k+1}\right)^{\frac{1}{\beta}}$
$(e^{-s/\alpha}x^{-\beta} + e^{-s/\alpha} - 1)^{\frac{-1}{\beta}}$	$(e^{-s/\alpha}(\lambda^{-\beta} - 1) + 1)^{\frac{-1}{\beta}}$	$\alpha\log(\lambda^{\beta}(e^{s/\alpha} - 1) + 1)$	$\alpha\log(ks^{\beta} + 1)$	$\left(\frac{\lambda^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}}$
$(1 - e^{s/\alpha} + e^{s/\alpha}x^{-\beta})^{\frac{-1}{\beta}}$	$(e^{s/\alpha}(\lambda^{-\beta} - 1) + 1)^{\frac{-1}{\beta}}$	$-\alpha\log(\lambda^{\beta}(e^{-s/\alpha} - 1) + 1)$	$-\alpha\log(ks^{\beta} + 1)$	$\left(\frac{\lambda^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}}$
$(1 - e^{-s/\alpha} + e^{-s/\alpha}x^{\beta})^{\frac{1}{\beta}}$	$(e^{-s/\alpha}(\lambda^{\beta} - 1) + 1)^{\frac{1}{\beta}}$	$\alpha\log(\lambda^{-\beta}(e^{s/\alpha} - 1) + 1)$	$\alpha\log(ks^{-\beta} + 1)$	$\left(\frac{\lambda^{\beta} + k}{k+1}\right)^{\frac{1}{\beta}}$
$(1 - e^{s/\alpha} + e^{s/\alpha}x^{\beta})^{\frac{1}{\beta}}$	$(e^{s/\alpha}(\lambda^{\beta} - 1) + 1)^{\frac{1}{\beta}}$	$-\alpha\log(\lambda^{-\beta}(e^{-s/\alpha} - 1) + 1)$	$-\alpha\log(ks^{-\beta} + 1)$	$\left(\frac{\lambda^{\beta} + k}{k+1}\right)^{\frac{1}{\beta}}$
$(1 - e^{-s/\alpha} + e^{-s/\alpha}x^{-\beta})^{\frac{-1}{\beta}}$	$(e^{-s/\alpha}(\lambda^{-\beta} - 1) + 1)^{\frac{-1}{\beta}}$	$\alpha\log(\lambda^{\beta}(e^{s/\alpha} - 1) + 1)$	$\alpha\log(ks^{\beta} + 1)$	$\left(\frac{\lambda^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}}$

3 Toward solving Equation (15)

Without loss of generality, hereafter we assume that $T(1) = 1$. Recall that Equation (15) is

$$H\left(\frac{T(\lambda v)}{T(v)}u\right) + H(\lambda v) = H\left(\frac{T(\lambda u)}{T(u)}v\right) + H(\lambda u)\,.$$

Setting $K = e^H$ and rearranging, we see that (15) implies

$$\frac{K\left(\frac{T(\lambda v)}{T(v)}u\right)}{K(\lambda u)} = \frac{K\left(\frac{T(\lambda u)}{T(u)}v\right)}{K(\lambda v)}\,. \tag{16}$$

Here are some thoughts about (16).

1. $K, T :]0,\infty[\to]0,\infty[$ are continuous and strictly monotonic. We have $\gamma(\lambda, H(v)) = \frac{T(\lambda v)}{T(v)}$ for all $\lambda, v > 0$, so $\frac{T(\lambda v)}{T(v)}$ is strictly monotonic in v for all fixed $\lambda \neq 1$. Note also that $\frac{T(\lambda v)}{T(v)}$ is strictly monotonic in λ for all fixed v because T is strictly monotonic, which is consistent with the fact that we are assuming $\gamma(\lambda, s)$ varies monotonically with λ.

2. A fixed point of T is an input λ_f such that $T(\lambda_f) = \lambda_f$. If λ_f is such a fixed point, then letting $\lambda = \lambda_f$ and $u = 1$ in (16), we see that $\frac{T(\lambda_f v)}{T(v)} = \lambda_f$, that is, $T(\lambda_f v) = \lambda_f T(v)$ for all $v > 0$. It is straightforward to use this to show that if λ_f is a fixed point of T, then $(\lambda_f)^n$ is also a fixed point of T for each integer n. We know that 1 is a fixed point of T; if there is also some other fixed point of T, then there must be infinitely many fixed points, T must be unbounded (above), and $\lim_{v\to 0} T(v)$ must equal 0. However, we know that 1 is the only fixed point of T: Since $\gamma(\lambda, H(v)) = \frac{T(\lambda v)}{T(v)}$, and since for any fixed point λ_0 of T we have $\frac{T(\lambda_0 v)}{T(v)} = \lambda_0$ (this is true because of (16)), it would have to be that $\gamma(\lambda_0, H(v)) = \lambda_0$ for all v, meaning $H(v) = 0$ for all v, which is possible only when $\lambda_0 = 1$. So, 1 is the only fixed point of T.

3. It seems that a natural thing to use/investigate is *regular variation* (Bingham et al., 1987; de Haan and Ferreira, 2006; Dzhafarov, 2002): A measurable function L :$]0,\infty[\to]0,\infty[$ varies regularly at 0 iff $\lim_{v\to 0}\frac{L(\lambda v)}{L(v)} = g(\lambda)$ for some function g, and for all $\lambda > 0$.[3] We will assume that either K or T varies regularly at 0.

[3] It turns out that in such a case, necessarily $g(\lambda) = \lambda^\rho$ for some real number ρ.

4. There is also the notion of regular variation at ∞: A measurable function $L :]0,\infty[\to]0,\infty[$ varies regularly at ∞ iff $\lim_{v\to\infty}\frac{L(\lambda v)}{L(v)} = h(\lambda)$ for some function h, and for all $\lambda > 0$.[4]

Attempting to make progress on (16), we utilize some well-known results.

Proposition 1 *Let $f :]0,\infty[\to]0,\infty[$ be measurable.*
(i) f varies regularly at 0 if and only if

$$\lim_{v\to 0}\frac{f(bv)}{f(cv)} = h\left(\frac{b}{c}\right) \quad \textit{for some function } h\textit{, and for all } b,c>0\,.$$

Moreover, it must be that $h(\lambda)=\lambda^{\tau}$ for some τ, and for all $\lambda>0$.
(ii) Define $\tilde{f}(x) = f\left(\frac{1}{x}\right)$ for all $x>0$. Then f is regularly varying at 0 with index ρ if and only if $\tilde{f}$ is regularly varying at infinity with index $-\rho$.
(iii) If f is regularly varying at ∞ with index ι, then

$$\lim_{\lambda\to\infty} f(\lambda) = \begin{cases}\infty, & \textit{if } \iota>0\\ 0, & \textit{if } \iota<0\,.\end{cases}$$

(The limit can be finite and nonzero only when $\iota = 0$.)
(iv) If f is regularly varying at 0 with index ρ, then

$$\lim_{\lambda\to 0} f(\lambda) = \begin{cases}0, & \textit{if } \rho>0\\ \infty, & \textit{if } \rho<0\,.\end{cases}$$

(The limit can be finite and nonzero only when $\rho = 0$.)

Proof. Let $f :]0,\infty[\to]0,\infty[$ be measurable.
(i) ($\Rightarrow$) Suppose f varies regularly at 0. Let b, c be positive real numbers. Then

$$\lim_{v\to 0}\frac{f(bv)}{f(cv)} = \lim_{\frac{v}{c}\to 0}\frac{f\left(b\frac{v}{c}\right)}{f(c\frac{v}{c})} = \lim_{\frac{v}{c}\to 0}\frac{f\left(\frac{b}{c}v\right)}{f(v)} = h\left(\frac{b}{c}\right) \quad \text{for some function } h\,,$$

where the last equality uses the fact that f varies regularly at 0.

($\Leftarrow$) Suppose $\lim_{v\to 0}\frac{f(bv)}{f(cv)} = h\left(\frac{b}{c}\right)$ for some function h, for all $b,c>0$. Setting $c=1$, we see that f varies regularly at 0.

The "Moreover" statement follows directly from footnote 3.
(ii) This is from page 18 of Bingham et al. (1987).

[4]It turns out that in such a case, necessarily $h(\lambda)=\lambda^{\tau}$ for some real number τ.

(iii) This is from page 22 of Bingham et al. (1987).
(iv) This is an immediate consequence of (ii) and (iii) of this proposition.
□

We have the following results about the K and T that satisfy (16).

Theorem 2 *Suppose* $K, T \ :]0, \infty[\to]0, \infty[$ *are continuous and satisfy* (16)*, with* K *strictly monotonic and* $T(1) = 1$.
(i) T *varies regularly at* 0 *if and only if* K *does.*
(ii) If either K *or* T *varies regularly at* 0*, then*

$$\frac{K(\lambda^\tau u)\lambda^\delta}{K(\lambda u)} = \left(\frac{T(\lambda u)}{T(u)}\right)^\delta \tag{17}$$

for some real numbers δ *and* τ*, and for all* $\lambda, u > 0$.
(iii) If either K *or* T *varies regularly at* 0*, then*

$$\frac{K(\lambda^\tau)\lambda^\delta}{K(\lambda)} = T(\lambda)^\delta \tag{18}$$

for some real numbers δ *and* τ*, and for all* $\lambda > 0$.
(iv) (K, T) *is a solution to* (16) *iff* (cK^ρ, T) *is a solution to* (16)*, for any* $c, \rho \neq 0$.
(v) Define $\tilde{K}(x) = K\left(\frac{1}{x}\right)$ *and* $\tilde{T}(x) = T\left(\frac{1}{x}\right)$ *for all* $x > 0$*. Then* (K, T) *is a solution to* (16) *if and only if* $\left(\tilde{K}, 1/\tilde{T}\right)$ *is a solution to* (16)*.*

Proof.
(i) ($\Rightarrow$) Assuming T varies regularly at 0, we have that $\lim_{v \to 0} \frac{T(\lambda v)}{T(v)} = \lambda^\tau$ for all $\lambda > 0$, for some real number τ. Since K is continuous, we can take the limit of the l.h.s. of (16) as $v \to 0$ to get

$$\lim_{v \to 0} \frac{K\left(\frac{T(\lambda v)}{T(v)} u\right)}{K(\lambda u)} = \frac{K\left(\lim_{v \to 0} \frac{T(\lambda v)}{T(v)} u\right)}{K(\lambda u)} = \frac{K\left(\lambda^\tau u\right)}{K(\lambda u)} .$$

So, we must have that the limit of the r.h.s. of (16) as $v \to 0$ exists and equals $\frac{K(\lambda^\tau u)}{K(\lambda u)}$, that is,

$$\lim_{v \to 0} \frac{K\left(\frac{T(\lambda u)}{T(u)} v\right)}{K(\lambda v)} = \frac{K\left(\lambda^\tau u\right)}{K(\lambda u)} . \tag{19}$$

Setting $u = 1$ and using the fact that $T(1) = 1$, we get from (19) that

$$\lim_{v \to 0} \frac{K\left(T(\lambda)v\right)}{K(\lambda v)} = \frac{K\left(\lambda^{\tau}\right)}{K(\lambda)}\,, \tag{20}$$

and using Lemma 1, we see that K varies regularly at 0.

($\Leftarrow$) Assuming K varies regularly at 0, we can set $u = 1$ in (16) and take the limit of the r.h.s. as $v \to 0$ to get

$$\lim_{v \to 0} \frac{K\left(T(\lambda)v\right)}{K(\lambda v)} = \left(\frac{T(\lambda)}{\lambda}\right)^{\delta}$$

for some δ, by Lemma 1. When we take the limit of the l.h.s. of (16) as $v \to 0$ (with $u = 1$), we must also get $\left(\frac{T(\lambda)}{\lambda}\right)^{\delta}$, so

$$\left(\frac{T(\lambda)}{\lambda}\right)^{\delta} = \lim_{v \to 0} \frac{K\left(\frac{T(\lambda v)}{T(v)}\right)}{K(\lambda)} = \frac{K\left(\lim_{v\to 0}\frac{T(\lambda v)}{T(v)}\right)}{K(\lambda)}\,, \tag{21}$$

where the second equality in (21) uses the fact that K is continuous. From (21) and the fact that K is strictly monotonic, we see that $\lim_{v\to 0}\frac{T(\lambda v)}{T(v)}$ depends only on λ, that is, T varies regularly at 0.

(ii) By (i) above, both K and T must vary regularly at 0 if either one does. In such a case, we must have $\lim_{v\to 0}\frac{T(\lambda v)}{T(v)} = \lambda^{\tau}$ for some τ and for all $\lambda > 0$. Taking the limit as $v \to 0$ on both sides of (16) and using (i) gives (17).

(iii) Apply (ii) above with $u = 1$.

(iv) Obvious, based on the form of (16).

(v) Recall again Equation (16):

$$\frac{K\left(\frac{T(\lambda v)}{T(v)}u\right)}{K(\lambda u)} = \frac{K\left(\frac{T(\lambda u)}{T(u)}v\right)}{K(\lambda v)}\,.$$

Defining $\tilde{T}(x) := T(1/x)$ for all $x \in]0,\infty[$, we get

$$\frac{K\left(\frac{\tilde{T}(\frac{1}{\lambda v})}{\tilde{T}(\frac{1}{v})}u\right)}{K(\lambda u)} = \frac{K\left(\frac{\tilde{T}(\frac{1}{\lambda u})}{\tilde{T}(\frac{1}{u})}v\right)}{K(\lambda v)}\,.$$

And defining $\tilde{K}(x) := K(1/x)$ for all $x \in]0,\infty[$ gives the following:

$$\frac{\tilde{K}\left(\frac{\tilde{T}(\frac{1}{v})}{\tilde{T}(\frac{1}{\lambda v})u}\right)}{\tilde{K}(\frac{1}{\lambda u})} = \frac{\tilde{K}\left(\frac{\tilde{T}(\frac{1}{u})}{\tilde{T}(\frac{1}{\lambda u})v}\right)}{\tilde{K}(\frac{1}{\lambda v})}\,.$$

Then, with $\tilde{u} := 1/u$, $\tilde{v} := 1/v$, and $\tilde{\lambda} := 1/\lambda$, we have

$$\frac{\tilde{K}\left(\frac{\tilde{T}(\tilde{v})}{\tilde{T}(\tilde{\lambda}\tilde{v})}\tilde{u}\right)}{\tilde{K}(\tilde{\lambda}\tilde{u})} = \frac{\tilde{K}\left(\frac{\tilde{T}(\tilde{u})}{\tilde{T}(\tilde{\lambda}\tilde{u})}\tilde{v}\right)}{\tilde{K}(\tilde{\lambda}\tilde{v})}\,. \tag{22}$$

Comparing (16) and (22), we get the result in (v).
□

Remark 3 *Looking at the solutions 15(a)-(f) in Iverson (2006), we have the following (arbitrarily choosing the solution in 15(a) to be $(K(x), T(x))$).*

15(a) and 15(e): $(K(x), T(x))$ is the solution
15(b) and 15(f): $\left(\frac{1}{K(\frac{1}{x})}, \frac{1}{T(\frac{1}{x})}\right)$ is the solution
15(c): $\left(K(\frac{1}{x}), \frac{1}{T(\frac{1}{x})}\right)$ is the solution
15(d): $\left(\frac{1}{K(x)}, T(x)\right)$ is the solution

We also see from Iverson (2006) that the values of δ and τ are limited. Theorem 6 below gets at how δ and τ are limited. First, we introduce a lemma from Chung (1975).

Lemma 4 (Chung (1975), Lemma 2.3.2) *Suppose that $f\ :]0,\infty[\to]0,\infty[$ is a non-decreasing function and the following condition holds for a sequence $\{a_n\}$ of positive numbers with $a_n \to 0$ as $n \to \infty$:*

$$\lim_{n\to\infty} \frac{f(a_n x)}{f(a_n)} = x^c \qquad \textit{where } c \in]0,\infty[\,. \tag{23}$$

Let $\{a_n{}'\}$ be a sequence of positive numbers. If

$$\lim_{n\to\infty} \frac{a_n{}'}{a_n} = \beta \tag{24}$$

with $\beta \in [0,\infty]$, then

$$\lim_{n\to\infty} \frac{f(a_n{}')}{f(a_n)} = \beta^c\,. \tag{25}$$

Conversely, if (25) *holds with $\beta \in [0,\infty]$, then* (24) *is true. (The analogous result for a non-increasing function also holds.)*

Remark 5 *Suppose that* $g :]0,\infty[\to]0,\infty[$ *is a non-increasing function and the following condition holds for a sequence* $\{a_n\}$ *of positive numbers with* $a_n \to 0$ *as* $n \to \infty$*:*

$$\lim_{n\to\infty} \frac{g(a_n x)}{g(a_n)} = x^c \qquad \textit{where } c \in]-\infty, 0[\,. \tag{26}$$

Let $\{a_n{}'\}$ *be a sequence of positive numbers. If*

$$\lim_{n\to\infty} \frac{a_n{}'}{a_n} = \beta \tag{27}$$

with $\beta \in [0,\infty]$*, then*

$$\lim_{n\to\infty} \frac{g(a_n{}')}{g(a_n)} = \beta^c\,. \tag{28}$$

Theorem 6 *Suppose* $K, T :]0,\infty[\to]0,\infty[$ *are continuous, strictly monotonic, and satisfy* (16)*. Suppose* K *varies regularly at* 0 *with index* δ*. Then by Theorem 2(i),* T *also varies regularly at* 0*; let its index be* τ*. We have the following.*

(i) If $\delta = 0$*, then* $\tau = 1$*;*

(ii) If $\delta \neq 0$ *and* $\tau > 0$*, then* $T(\lambda) = \lambda^\tau$ *and* $K(\lambda) = K(1)\lambda^\delta$ *for all* $\lambda > 0$*;*

(iii) If $\delta \neq 0$*,* T *is strictly increasing and* $\tau = 0$*, then* $\lim_{\lambda\to 0} T(\lambda) = \ell$*, where* $0 < \ell < \infty$*.*

Proof. From Theorem 2(iii), we have

$$\frac{K(\lambda^\tau)\lambda^\delta}{K(\lambda)} = T(\lambda)^\delta$$

for all $\lambda > 0$. If $\delta = 0$, then clearly $\tau = 1$ because K is strictly monotonic. This means (i) holds.

For (ii), suppose that $\tau > 0$. Suppose first that $\delta > 0$. Then $\lim_{\lambda\to 0} K(\lambda) = 0$ by Proposition 1(iv), so K, already assumed strictly monotonic, must be strictly increasing. Since K is regularly varying at 0, we get that Condition (23) applies with $f = K$ (for *every* such sequence $\{a_n\}$). Consider (16), slightly rewritten as follows:

$$\frac{K(\lambda v)}{K(\lambda u)} = \frac{K\left(\frac{T(\lambda u)}{T(u)} v\right)}{K\left(\frac{T(\lambda v)}{T(v)} u\right)}\,. \tag{29}$$

If $\tau > 0$, it must be that $\lim_{\lambda \to 0} T(\lambda) = 0$, again by Proposition 1(iv). Let $\{\lambda_n\}$ be a sequence of positive reals such that $\lim_{n\to\infty} \lambda_n = 0$. Let ${a_n}' = \frac{T(\lambda_n u)}{T(u)} v$ and $a_n = \frac{T(\lambda_n v)}{T(v)} u$ for each positive integer n. Then $a_n \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{{a_n}'}{a_n} = \lim_{n\to\infty} \frac{\frac{T(\lambda_n u)}{T(u)} v}{\frac{T(\lambda_n v)}{T(v)} u} = \frac{v}{u} \frac{T(v)}{T(u)} \lim_{n\to\infty} \frac{T(\lambda_n u)}{T(\lambda_n v)} = \frac{v}{u} \frac{T(v)}{T(u)} \left(\frac{u}{v}\right)^{\tau} ,$$

where the last equality uses the fact that T is regularly varying at 0. From Lemma 4, then, we have

$$\lim_{n\to\infty} \frac{K({a_n}')}{K(a_n)} = \lim_{n\to\infty} \frac{K\left(\frac{T(\lambda_n u)}{T(u)} v\right)}{K\left(\frac{T(\lambda_n v)}{T(v)} u\right)} = \left(\frac{v}{u} \frac{T(v)}{T(u)} \left(\frac{u}{v}\right)^{\tau}\right)^{\delta} .$$

Now, (29) gives

$$\frac{K(\lambda_n v)}{K(\lambda_n u)} = \frac{K\left(\frac{T(\lambda_n u)}{T(u)} v\right)}{K\left(\frac{T(\lambda_n v)}{T(v)} u\right)}$$

for all n, and taking the limit of both sides as $n \to \infty$ gives

$$\left(\frac{v}{u}\right)^{\delta} = \left(\frac{v}{u} \frac{T(v)}{T(u)} \left(\frac{u}{v}\right)^{\tau}\right)^{\delta} . \tag{30}$$

Rearranging (30) and setting $u = 1$ gives $T(v) = v^{\tau}$. Using $\frac{K(\lambda^{\tau} u)\lambda^{\delta}}{K(\lambda u)} = \left(\frac{T(\lambda u)}{T(u)}\right)^{\delta}$ from Theorem 2(ii), with $T(v) = v^{\tau}$ we have

$$\frac{K(\lambda^{\tau} u)}{K(\lambda u)} = \left(\frac{\lambda^{\tau}}{\lambda}\right)^{\delta} ,$$

and setting $u = \frac{1}{\lambda}$ gives $K(\lambda^{\tau-1}) = K(1)\left(\lambda^{\tau-1}\right)^{\delta}$, i.e., $K(\lambda) = K(1)\lambda^{\delta}$. (Note that this derivation does not hold for $\tau = 1$. Indeed, $\tau = 1$ also does not work regarding the condition mentioned several pages above that 1 is the only fixed point of T.) This completes (ii) of the theorem when δ is assumed greater than 0. For $\delta < 0$, the argument is similar, except K is strictly decreasing, and Remark 5 is used instead of Lemma 4.

For (iii), we proceed by contradiction. Suppose $\delta > 0$, T is strictly increasing and $\tau = 0$, but $\lim_{\lambda\to 0} T(\lambda) \neq \ell$ for any finite, positive ℓ. Because T is strictly increasing, we cannot have $\lim_{\lambda\to 0} T(\lambda) = \infty$. So it would have to be that $\lim_{\lambda\to 0} T(\lambda) = 0$. (Note that

$\lim_{\lambda\to 0} T(\lambda)$ must exist because T is monotone and bounded below.) Again using Lemma 4 and proceeding as for (ii), we get (30), that is,

$$\left(\frac{v}{u}\right)^{\delta} = \left(\frac{v}{u}\frac{T(v)}{T(u)}\left(\frac{u}{v}\right)^{\tau}\right)^{\delta}.$$

Since we are assuming $\tau = 0$, the above implies $T(v) = T(u)$ for all $u, v > 0$, which is impossible for strictly increasing T. So, it cannot be that $\lim_{\lambda\to 0} T(\lambda) = 0$, which means we must have $\lim_{\lambda\to 0} T(\lambda) = \ell$, where $0 < \ell < \infty$. Similarly, we derive a contradiction assuming $\lim_{\lambda\to 0} T(\lambda) \neq \ell$ for any finite, positive ℓ while also assuming $\delta < 0$, T is strictly increasing and $\tau = 0$. (This time, Remark 5 is used.) This completes the proof of (iii).
□

4 Two interesting cases: (i) $\delta = 0, \tau = 1$, (ii) $\delta \neq 0, \tau = 0$ in Theorem 6

We will assume that the solution K of (16) is regularly varying at 0 and that the index of variation is a. By Theorem 2(v), we have that $(K(x), T(x))$ is a solution iff $\left(K(\frac{1}{x}), \frac{1}{T(\frac{1}{x})}\right)$ is a solution. From Proposition 1(ii), since $K(x)$ is regularly varying at 0 with index a, we have that $K(\frac{1}{x})$ is regularly varying at infinity with index $-a$. Now, we assume $K(\frac{1}{x})$ is regularly varying at 0 because it is a solution of (16)—we are assuming each solution "K" (whatever its form) is regularly varying at 0. Call b the index of regular variation of $K(\frac{1}{x})$ at 0. Then again by Proposition 1(ii), $K(x)$ is regularly varying at infinity with index $-b$. Now for T. Since K is regularly varying at 0, so is T. We will call c the index of regular variation of T at 0. From Proposition 1(ii), since $T(x)$ is regularly varying at 0 with index c, we have that $T(\frac{1}{x})$ is regularly varying at infinity with index $-c$, so $\frac{1}{T(\frac{1}{x})}$ is regularly varying at infinity with index c. Since $K(\frac{1}{x})$ is regularly varying at 0, so is $\frac{1}{T(\frac{1}{x})}$ (by Theorem 2(v), because $\left(K(\frac{1}{x}), \frac{1}{T(\frac{1}{x})}\right)$ is a solution). Calling d the index of regular variation of $\frac{1}{T(\frac{1}{x})}$ at 0, we have from Proposition 1(ii) that $\frac{1}{T(x)}$ is regularly varying at infinity with index $-d$, so that $\frac{1}{T(\frac{1}{x})}$ is regularly varying at 0 with index d.

Let us distinguish τ_∞, δ_∞ defined by

$$\lambda^{\tau_\infty} = \lim_{u\to\infty} \frac{T(\lambda u)}{T(u)} \qquad \text{and} \qquad \lambda^{\delta_\infty} = \lim_{u\to\infty} \frac{K(\lambda u)}{K(u)}$$

from τ_0, δ_0 defined by

$$\lambda^{\tau_0} = \lim_{u\to 0} \frac{T(\lambda u)}{T(u)} \qquad \text{and} \qquad \lambda^{\delta_0} = \lim_{u\to 0} \frac{K(\lambda u)}{K(u)} \,.$$

The cases we have narrowed down to are (i) $\delta_0 = 0$, $\tau_0 = 1$ and (ii) $\delta_0 \neq 0$, $\tau_0 = 0$. It appears τ_∞ and τ_0 are related to the solutions in Iverson (2006), and δ_∞ and δ_0 are related as well—for each solution (K, T), it appears that $(\delta_\infty, \tau_\infty) = (nonzero, 0)$ iff $(\delta_0, \tau_0) = (0, 1)$, and also $(\delta_\infty, \tau_\infty) = (0, 1)$ iff $(\delta_0, \tau_0) = (nonzero, 0)$. (See below.) Further, it appears that the solutions in Iverson (2006) come in the forms $(K(x), T(x))$, $\left(\frac{1}{K(\frac{1}{x})}, \frac{1}{T(\frac{1}{x})}\right)$, $\left(K(\frac{1}{x}), \frac{1}{T(\frac{1}{x})}\right)$, and $\left(\frac{1}{K(x)}, T(x)\right)$. We note that these forms of solutions are consistent with the forms $(cK(x)^\rho, T(x))$ and $\left(cK(\frac{1}{x})^\rho, \frac{1}{T(\frac{1}{x})}\right)$ of Theorem 2(iv) and (v).

Can we conclude by some sort of argument that finding solutions for only half the cases—e.g., finding the solutions only for the case $(\delta_0, \tau_0) = (nonzero, 0)$—will automatically give us all the solutions? To get some insights, in the following we summarize the solutions in Iverson (2006) to see how they are related to δ and τ in Theorem 6.

All of the forms of T look like either $T(\lambda) = \left(\frac{\lambda^\beta + k}{k+1}\right)^{\frac{1}{\beta}}$ or $T(\lambda) = \left(\frac{\lambda^{-\beta} + k}{k+1}\right)^{\frac{1}{-\beta}}$, where $\beta > 0$, and all of the forms of K look like $K(s) = (ks^{\pm\beta} + 1)^{\pm\alpha}$ (that is, $K(s) = (ks^\beta + 1)^\alpha$, $K(s) = (ks^\beta + 1)^{-\alpha}$, $K(s) = (ks^{-\beta} + 1)^\alpha$, or $K(s) = (ks^{-\beta} + 1)^{-\alpha}$), where $\alpha, \beta > 0$.

15(a), 15(e): $K(x) = (kx^{-\beta} + 1)^{-\alpha}$ **and** $T(x) = \left(\frac{x^\beta + k}{k+1}\right)^{\frac{1}{\beta}}$

$$\lim_{u\to 0} \frac{K(\lambda u)}{K(u)} = \lim_{u\to 0} \frac{(k(\lambda u)^{-\beta} + 1)^{-\alpha}}{(ku^{-\beta} + 1)^{-\alpha}} = \lim_{u\to 0} \frac{(k\lambda^{-\beta} + u^\beta)^{-\alpha}}{(k + u^\beta)^{-\alpha}} = \lambda^{\alpha\beta}$$

$$\lim_{u\to \infty} \frac{K(\lambda u)}{K(u)} = \lim_{u\to \infty} \frac{(k(\lambda u)^{-\beta} + 1)^{-\alpha}}{(ku^{-\beta} + 1)^{-\alpha}} = 1$$

$$\lim_{u\to 0} \frac{T(\lambda u)}{T(u)} = \lim_{u\to 0} \left(\frac{(\lambda u)^\beta + k}{k+1}\right)^{\frac{1}{\beta}} \cdot \left(\frac{k+1}{u^\beta + k}\right)^{\frac{1}{\beta}} = 1$$

$$\lim_{u\to \infty} \frac{T(\lambda u)}{T(u)} = \lim_{u\to \infty} \left(\frac{(\lambda u)^\beta + k}{k+1}\right)^{\frac{1}{\beta}} \cdot \left(\frac{k+1}{u^\beta + k}\right)^{\frac{1}{\beta}} = \lim_{u\to \infty} \left(\frac{(\lambda u)^\beta + k}{u^\beta + k}\right)^{\frac{1}{\beta}} = \lim_{u\to \infty} \left(\frac{\lambda^\beta + k/u^\beta}{1 + k/u^\beta}\right)^{\frac{1}{\beta}} = \lambda$$

- $\delta_0 = \alpha\beta$ and $\delta_\infty = 0$
- $\tau_0 = 0$ and $\tau_\infty = 1$
- $\lim_{s\to 0} K(s) = 0$ and $\lim_{s\to\infty} K(s) = 1$

- $\lim_{\lambda\to 0} T(\lambda) = \left(\frac{k}{k+1}\right)^{\frac{1}{\beta}}$ and $\lim_{\lambda\to\infty} T(\lambda) = \infty$

15(b), 15(f): $K(x) = (kx^\beta + 1)^\alpha$ **and** $T(x) = \left(\frac{x^{-\beta}+k}{k+1}\right)^{\frac{-1}{\beta}}$

$$\lim_{u\to 0} \frac{K(\lambda u)}{K(u)} = \lim_{u\to 0} \frac{(k(\lambda u)^\beta + 1)^\alpha}{(ku^\beta + 1)^\alpha} = 1$$

$$\lim_{u\to\infty} \frac{K(\lambda u)}{K(u)} = \lim_{u\to\infty} \frac{(k(\lambda u)^\beta + 1)^\alpha}{(ku^\beta + 1)^\alpha} = \lim_{u\to\infty} \frac{(k\lambda^\beta + 1/u^\beta)^\alpha}{(k + 1/u^\beta)^\alpha} = \lambda^{\alpha\beta}$$

$$\lim_{u\to 0} \frac{T(\lambda u)}{T(u)} = \lim_{u\to 0} \left(\frac{(\lambda u)^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}} \cdot \left(\frac{k+1}{u^{-\beta} + k}\right)^{\frac{-1}{\beta}} = \lim_{u\to 0} \left(\frac{\lambda^{-\beta} + ku^\beta}{1 + ku^\beta}\right)^{\frac{-1}{\beta}} = \lambda$$

$$\lim_{u\to\infty} \frac{T(\lambda u)}{T(u)} = \lim_{u\to\infty} \left(\frac{(\lambda u)^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}} \cdot \left(\frac{k+1}{u^{-\beta} + k}\right)^{\frac{-1}{\beta}} = 1$$

- $\delta_0 = 0$ and $\delta_\infty = \alpha\beta$
- $\tau_0 = 1$ and $\tau_\infty = 0$
- $\lim_{s\to 0} K(s) = 1$ and $\lim_{s\to\infty} K(s) = \infty$
- $\lim_{\lambda\to 0} T(\lambda) = 0$ and $\lim_{\lambda\to\infty} T(\lambda) = \left(\frac{k}{k+1}\right)^{\frac{-1}{\beta}}$

15(c): $K(x) = (kx^\beta + 1)^{-\alpha}$ **and** $T(x) = \left(\frac{x^{-\beta}+k}{k+1}\right)^{\frac{-1}{\beta}}$

$$\lim_{u\to 0} \frac{K(\lambda u)}{K(u)} = \lim_{u\to 0} \frac{(k(\lambda u)^\beta + 1)^{-\alpha}}{(ku^\beta + 1)^{-\alpha}} = 1$$

$$\lim_{u\to\infty} \frac{K(\lambda u)}{K(u)} = \lim_{u\to\infty} \frac{(k(\lambda u)^\beta + 1)^{-\alpha}}{(ku^\beta + 1)^{-\alpha}} = \lim_{u\to\infty} \frac{(k\lambda^\beta + 1/u^\beta)^{-\alpha}}{(k + 1/u^\beta)^{-\alpha}} = \lambda^{-\alpha\beta}$$

$$\lim_{u\to 0} \frac{T(\lambda u)}{T(u)} = \lim_{u\to 0} \left(\frac{(\lambda u)^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}} \cdot \left(\frac{k+1}{u^{-\beta} + k}\right)^{\frac{-1}{\beta}} = \lim_{u\to 0} \left(\frac{\lambda^{-\beta} + ku^\beta}{1 + ku^\beta}\right)^{\frac{-1}{\beta}} = \lambda$$

$$\lim_{u\to\infty} \frac{T(\lambda u)}{T(u)} = \lim_{u\to\infty} \left(\frac{(\lambda u)^{-\beta} + k}{k+1}\right)^{\frac{-1}{\beta}} \cdot \left(\frac{k+1}{u^{-\beta} + k}\right)^{\frac{-1}{\beta}} = 1$$

- $\delta_0 = 0$ and $\delta_\infty = -\alpha\beta$
- $\tau_0 = 1$ and $\tau_\infty = 0$

- $\lim_{s\to 0} K(s) = 1$ and $\lim_{s\to\infty} K(s) = 0$
- $\lim_{\lambda\to 0} T(\lambda) = 0$ and $\lim_{\lambda\to\infty} T(\lambda) = \left(\frac{k}{k+1}\right)^{\frac{-1}{\beta}}$

15(d): $K(x) = (kx^{-\beta}+1)^{\alpha}$ **and** $T(x) = \left(\frac{x^{\beta}+k}{k+1}\right)^{\frac{1}{\beta}}$

$$\lim_{u\to 0}\frac{K(\lambda u)}{K(u)} = \lim_{u\to 0}\frac{(k(\lambda u)^{-\beta}+1)^{\alpha}}{(ku^{-\beta}+1)^{\alpha}} = \lim_{u\to 0}\frac{(k\lambda^{-\beta}+u^{\beta})^{\alpha}}{(k+u^{\beta})^{\alpha}} = \lambda^{-\alpha\beta}$$

$$\lim_{u\to\infty}\frac{K(\lambda u)}{K(u)} = \lim_{u\to\infty}\frac{(k(\lambda u)^{-\beta}+1)^{\alpha}}{(ku^{-\beta}+1)^{\alpha}} = 1$$

$$\lim_{u\to 0}\frac{T(\lambda u)}{T(u)} = \lim_{u\to 0}\left(\frac{(\lambda u)^{\beta}+k}{k+1}\right)^{\frac{1}{\beta}}\cdot\left(\frac{k+1}{u^{\beta}+k}\right)^{\frac{1}{\beta}} = 1$$

$$\lim_{u\to\infty}\frac{T(\lambda u)}{T(u)} = \lim_{u\to\infty}\left(\frac{(\lambda u)^{\beta}+k}{k+1}\right)^{\frac{1}{\beta}}\cdot\left(\frac{k+1}{u^{\beta}+k}\right)^{\frac{1}{\beta}} = \lim_{u\to\infty}\left(\frac{(\lambda u)^{\beta}+k}{u^{\beta}+k}\right)^{\frac{1}{\beta}} = \lim_{u\to\infty}\left(\frac{\lambda^{\beta}+k/u^{\beta}}{1+k/u^{\beta}}\right)^{\frac{1}{\beta}} = \lambda$$

- $\delta_0 = -\alpha\beta$ and $\delta_\infty = 0$
- $\tau_0 = 0$ and $\tau_\infty = 1$
- $\lim_{s\to 0} K(s) = \infty$ and $\lim_{s\to\infty} K(s) = 1$
- $\lim_{\lambda\to 0} T(\lambda) = \left(\frac{k}{k+1}\right)^{\frac{1}{\beta}}$ and $\lim_{\lambda\to\infty} T(\lambda) = \infty$

5 Further thoughts

Note the result in Theorem 6(iii) that $\lim_{\lambda\to 0} T(\lambda) = \ell$, where $0 < \ell < \infty$. Let us assume it, with $\tau = 0$ and see what we can derive. Consider again (29):

$$\frac{K\left(\frac{T(\lambda u)}{T(u)}v\right)}{K\left(\frac{T(\lambda v)}{T(v)}u\right)} = \frac{K(\lambda v)}{K(\lambda u)}\,.$$

Assuming $0 < \lim_{\lambda\to\infty} T(\lambda) = \ell < \infty$, taking $\lim_{\lambda\to\infty}$ of both sides gives

$$\frac{K\left(\frac{\ell}{T(v)}u\right)}{K\left(\frac{\ell}{T(u)}v\right)} = \left(\frac{u}{v}\right)^{\delta}\,,$$

which is the same thing as

$$\frac{K\left(\frac{\ell}{T(v)}u\right)^{1/\delta}}{K\left(\frac{\ell}{T(u)}v\right)^{1/\delta}} = \frac{u}{v}\,,$$

and remembering that $T(\lambda) = \left(\frac{K(1)}{K(\lambda)}\right)^{1/\delta} \lambda$ (which is from Theorem 2(iii) with $\tau = 0$), we have

$$\frac{K\left(\ell\frac{K(v)^{1/\delta}}{K(1)^{1/\delta}} \cdot \frac{u}{v}\right)^{1/\delta}}{K\left(\ell\frac{K(u)^{1/\delta}}{K(1)^{1/\delta}} \cdot \frac{v}{u}\right)^{1/\delta}} = \frac{u}{v}\,.$$

Now, with $w = \frac{u}{v}$ and $\tilde{M}(x) = K(x)^{1/\delta}$, we get

$$\frac{\tilde{M}\left(\ell\frac{\tilde{M}(v)}{\tilde{M}(1)} \cdot w\right)}{\tilde{M}\left(\ell\frac{\tilde{M}(wv)}{\tilde{M}(1)} \cdot \frac{1}{w}\right)} = w\,,$$

and with $z = \frac{\tilde{M}(1)v}{\ell}$, $M(x) = \tilde{M}(\frac{\ell x}{\tilde{M}(1)})$, we have

$$\frac{M\left(M(z)w\right)}{M\left(M(wz) \cdot \frac{1}{w}\right)} = w\,,$$

that is,

$$\frac{M\left(M(z)w\right)}{w} = M\left(\frac{M(wz)}{w}\right)\,, \tag{31}$$

which holds for all $w, z > 0$. Maybe we could take good advantage of (31). One property that looks good for it is the following: Replacing z with $M(z)$ gives

$$\begin{aligned}
\frac{M\left(M^2(z)w\right)}{w} &= M\left(\frac{M(wM(z))}{w}\right) \\
&= M\left(M\left(\frac{M(wz)}{w}\right)\right) \\
&= M^2\left(\frac{M(wz)}{w}\right)\,,
\end{aligned}$$

and continuing in this way, we get for all positive integers n,

$$\frac{M\left(M^n(z)w\right)}{w} = M^n\left(\frac{M(wz)}{w}\right). \tag{32}$$

This property of (31) is appealing, as it says that applying M to $\frac{M(wz)}{w}$ is the same as simply applying M to z in $\frac{M(wz)}{w}$. Furthermore, we have that M is regularly varying since K is.

We wrap up the report with some final comments for future work.

- We have made good progress in getting to Equation (31):

  $$\frac{M\left(M(z)w\right)}{w} = M\left(\frac{M(wz)}{w}\right) \qquad (w, z > 0).$$

  We know of at least two solutions consistent with our various conditions, namely, the solutions $M(x) = kx$ for some nonzero constant k, and $M(x) = (1 + x^\beta)^{1/\beta}$ for some nonzero constant β. Proving that these are the only solutions, or finding all the other solutions, will be a challenge. It is very possible that either M has every point as a fixed point (namely $M(x) = x$ for all $x > 0$), or M has no fixed points. More work needs to be done if we are to solve (31).

- There are a number of properties of regularly varying functions that one may take advantage of. A summary of some properties is in de Haan and Ferreira (2006), pages 366+. Here is one of those properties:

  > If f is regularly varying with index $\alpha > 0$ ($\alpha < 0$) then f is asymptotically equivalent to a strictly increasing (decreasing) differentiable function g with derivative g', which is regularly varying with index $\alpha - 1$ if $\alpha > 0$, and $-g'$ is regularly varying with index $\alpha - 1$ if $\alpha < 0$.

  This may help us in solving Equation (31).

- We have been eliminating the differentiability assumption and instead assuming regular variation. It looks the regular variation assumption is weaker, that is, there are functions that satisfy regular variation (and are defined on $]0, \infty[$, are monotonic, continuous, etc.) but are not differentiable. It would be good to have an example and an intuition for the difference between regular variation and differentiability. Also, a natural next step would be to examine the relationship to Dzhafarov (2002)'s multidimensional Fechnerian work.

- Assume a Fechnerian representation $P(x, y) = F[u(x) - u(y)]$. Suppose some 'initial measurement unit' has been chosen for measuring the stimuli. Then for any $\lambda > 0$, where λ is the factor for converting the initial measurement unit to a new unit, we have $P_\lambda(\lambda x, \lambda y) = F_\lambda[u_\lambda(\lambda x) - u_\lambda(\lambda y)]$, where P_λ, F_λ and u_λ may depend on λ, and
$$P(x, y) = P_\lambda(\lambda x, \lambda y)\,. \tag{33}$$
This means that
$$F[u(x) - u(y)] = F_\lambda[u_\lambda(\lambda x) - u_\lambda(\lambda y)]\,. \tag{34}$$
That is,
$$u(\xi_s(x)) - u(x) = s = u_\lambda(\lambda\xi_s(x)) - u_\lambda(\lambda x)\,. \tag{35}$$
Define ${}_\lambda\xi_s(x)$ to be the intensity of the stimulus judged greater than x when the unit given by λ is being used (and the index is s). Then we have
$$\lambda\xi_s(x) = {}_\lambda\xi_s(\lambda x)\,, \tag{36}$$
which is a "meaningfulness constraint" (see Falmagne & Doble, 2015) relating the functions in the family $\mathcal{S} = \{\,{}_\lambda\xi_s \mid \lambda > 0\}$, where ${}_1\xi_s := \xi_s$. We also assume that the "initial code" ξ_s satisfies the law of similarity (Equation (1)), that is,
$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x)\,.$$
Following Falmagne and Doble (2015) this would mean that assuming (1) and (36), it would be the case that all of the members in $\mathcal{S}$ satisfy the law of similarity. Toward examining this, we see that, for any $\theta > 0$, using successively (36) and (1), we get
$${}_\lambda\xi_s(\theta x) = \lambda\,\xi_s\left(\frac{\theta}{\lambda}x\right) = \lambda\,\gamma\left(\frac{\theta}{\lambda}, s\right)\xi_{\eta\left(\frac{\theta}{\lambda},s\right)}(x)\,.$$
So we have somewhat of a law of similarity propagating, namely,
$${}_\lambda\xi_s(\theta x) = \lambda\,\gamma\left(\frac{\theta}{\lambda}, s\right)\xi_{\eta\left(\frac{\theta}{\lambda},s\right)}(x)\,, \tag{37}$$
which we can think of as
$${}_\lambda\xi_s(\theta x) = \gamma_\lambda\,(\theta, s)\,\xi_{\eta_\lambda(\theta,s)}(x)\,. \tag{38}$$
In (38), we could have that $\gamma_\lambda = \gamma$ and/or that $\eta_\lambda = \eta$, which would give a stronger propagation result than if these were not true. From Iverson (2006)'s solutions, it does not appear that $\gamma_\lambda = \gamma$ or that $\eta_\lambda = \eta$ for the interesting cases, (15a)-(15f), or for some of the other cases. It would be interesting to look at from the meaningfulness viewpoint whether something like (38) helps us in solving Equation (15).

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出國報告（出國類別：移地研究）

移地研究

服務機關：國立台灣大學心理系
姓名職稱：徐永豐 教授
派赴國家：Irvine, USA
出國期間：July 17–July 21, 2018
報告日期：October 30, 2018
經費來源：科技部

The main purpose of this trip was to meet Dr. Chris Doble to discuss our joint work on ‘regular variation’ in psychophysics. During my stay, we made some progress on using regularly varying functions to help narrow down the possible functional forms of the scales in the Fechnerian representation without assuming differentiability. Some of the results will be summarized in the final report of this project. We also briefly discussed a related work by Prof. Ehtibar Dzhafarov on regular variation to see what it means for Fechnerian scaling via the ‘law of similarity.’

The discussion with Dr. Doble was critical and beneficial, as after the trip I flew to Europe to present our work in the 2018 European Mathematical Psychology Group meeting.

On a side note, while in Southern California Dr. Doble and I also spent time discussing the work being done at ALEKS on modeling retention of mathematics learned via the ALEKS system (an adaptive learning and assessment system) to address the issues of forgetting curves and testing effect.

出國報告（出國類別：出席會議）

出席 2018 European Mathematical Psychology Group 年會

服務機關：國立台灣大學心理系
姓名職稱：徐永豐 教授
派赴國家：Genova, Italy
出國期間：July 30–August 2, 2018
報告日期：September 13, 2018
經費來源：科技部

目的

The purpose of this trip was to present my work in the 2018 Annual Meeting of the European Mathematical Psychology Group (EMPG), which began the late afternoon of July 30th and ended the afternoon of August 2nd in Genova, Italy. I also took a detour before and after the conference, flying to Irvine in Southern California, USA, to discuss the current project with Dr. Chris Doble.

過程

In the EMPG meeting I gave a poster presentation titled "Characterizing the law of similarity in psychophysics". Here is the abstract.

"Let $\xi_s(x) = x + \Delta_s(x)$ (where s represents the response criterion and Δ_s is the just noticeable difference) be the (Weber) sensitivities in psychophysics. Iverson (2006, *J Math Psych*, *50*, 283-289) introduced a *law of similarity* on ξ_s and studied its impact on the Fechnerian representation. This law states that

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda, s)}(x), \tag{1}$$

where $\gamma(\lambda, s)$ and $\eta(\lambda, s)$ are continuous in the two variables, $\gamma(\lambda, s)$ varies monotonically with λ, and $\eta(\lambda, s)$ varies monotonically with s. Following Hsu and Iverson (2016, *J Math Psych, 75*, 150-156) and imposing the notion of regular variation (Dzhafarov, 2002, *J Math Psych, 46*, 226-244) on some of the functions involved, in this study we attempt to characterize the forms of γ and η in (1) and link the results to the solutions in Iverson (2006)."

The feedback from several participants was informative; it gave me some insights into how the research could be expanded in the future.

心得

The EMPG meeting has much smaller size in attendance than the US-based annual meeting of the Society for Mathematical Psychology. Nonetheless, the EMPG meeting is more focused and provides more opportunities to interact intellectually with other scholars. For example, during this year's meeting I got a chance to meet Professor Karl Klauer after his keynote speech on extending the multinomial processing-tree (MPT) models to response times. I chatted with him, mentioning that I had applied his latent-class MPT model to clinical data in Taiwan several years ago. Some of other talks were inspiring too. For example, Professor Yutaka Matsushita gave an interesting extension of measurement theory to derive a hyperbolic-type of discounting function in intertemporal choice---a topic that I also have put some thought into recently. I chatted with him and we both are open to possible collaboration in the future. Overall, attending the EMPG meeting turned out to be a very rewarding experience for me.

106年度專題研究計畫成果彙整表

計畫主持人：徐永豐					計畫編號：106-2410-H-002-080-		
計畫名稱：心理物理學相似律所含參數之函數性質於費區納表徵下之探究							
成果項目					量化	單位	質化（說明：各成果項目請附佐證資料或細項說明，如期刊名稱、年份、卷期、起訖頁數、證號...等）
國內	學術性論文	期刊論文			0	篇	
		研討會論文			0		
		專書			0	本	
		專書論文			0	章	
		技術報告			0	篇	
		其他			0	篇	
	智慧財產權及成果	專利權	發明專利	申請中	0	件	
				已獲得	0		
			新型/設計專利		0		
		商標權			0		
		營業秘密			0		
		積體電路電路布局權			0		
		著作權			0		
		品種權			0		
		其他			0		
	技術移轉	件數			0	件	
		收入			0	千元	
國外	學術性論文	期刊論文			0	篇	
		研討會論文			1		I presented some of the results at the 2018 Annual Meeting of the European Mathematical Psychology Group held in Genova, Italy
		專書			0	本	
		專書論文			0	章	
		技術報告			0	篇	
		其他			0	篇	
	智慧財產權及成果	專利權	發明專利	申請中	0	件	
				已獲得	0		
			新型/設計專利		0		
		商標權			0		
		營業秘密			0		
		積體電路電路布局權			0		

		著作權	0		
		品種權	0		
		其他	0		
	技術移轉	件數	0	件	
		收入	0	千元	
參與計畫人力	本國籍	大專生	4	人次	主要乃提供支援，提供實作經驗。
		碩士生	0		
		博士生	0		
		博士後研究員	0		
		專任助理	0		
	非本國籍	大專生	0		
		碩士生	0		
		博士生	0		
		博士後研究員	0		
		專任助理	0		
其他成果 （無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。）					

科技部補助專題研究計畫成果自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現（簡要敘述成果是否具有政策應用參考價值及具影響公共利益之重大發現）或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估 ■達成目標 □未達成目標（請說明，以100字為限） 　　□實驗失敗 　　□因故實驗中斷 　　□其他原因 說明：
2. 研究成果在學術期刊發表或申請專利等情形（請於其他欄註明專利及技轉之證號、合約、申請及洽談等詳細資訊） 論文：□已發表　□未發表之文稿　■撰寫中　□無 專利：□已獲得　□申請中　■無 技轉：□已技轉　□洽談中　■無 其他：（以200字為限）
3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性，以500字為限） 本計畫屬(理論)心理物理學基礎研究，自有其學術貢獻，然對當今社會及經濟面向並無立即可見影響。
4. 主要發現 本研究具有政策應用參考價值：■否　□是，建議提供機關 （勾選「是」者，請列舉建議可提供施政參考之業務主管機關） 本研究具影響公共利益之重大發現：□否　□是 說明：（以150字為限）