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心理物理學相似律與仿射表徵互動之探究

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中 文 摘 要 ： 我們採用一藉區辨極限及差異閾積分建構心理物理量尺的「費區納法」，探究心理物理「仿射表徵」之量尺性質。我們假設一心理物理「相似律」，探討在此同質性假設下，仿射表徵的量尺函數形式。我們並探究在此框設下，仿射表徵可簡化為費區納表徵的條件，並討論其與Iverson (2006) 於費區納表徵下所得之結果的關聯。本研究成果已寫成論文 ” Conditions under which affine representations are also Fechnerian under the power law of similarity on the Weber sensitivities” ，且即將發表於Journal of Mathematical Psychology期刊（見附文）。

中 文 關 鍵 詞 ： 仿射表徵、平衡條件、費區納法、費區納表徵、相似律

英 文 摘 要 ： In this project we use the Fechner method (of taking the limit of sequence of integrals of jnd’ s) to construct the scales in a (weakly balanced) affine representation. We postulate a psychophysical law of similarity, and study its impact on the possible functional forms in the affine representation. Under this framework we further study the conditions under which affine representations are also Fechnerian, and link the results to the solutions in Iverson (2006) that was worked out within the Fechnerian framework. The work of this project has been summarized in a paper titled “Conditions under which affine representations are also Fechnerian under the power law of similarity on the Weber sensitivities” to be published in the Journal of Mathematical Psychology (see the attached file).

英 文 關 鍵 詞 ： affine representation; balance condition; Fechner method; Fechnerian representation; law of similarity

摘要

我們採用一藉區辨極限及差異閾積分建構心理物理量尺的「費區納法」，探究心理物理「仿射表徵」之量尺性質。我們假設一心理物理「相似律」，探討在此同質性假設下，仿射表徵的量尺函數形式。我們並探究在此框設下，仿射表徵可簡化為費區納表徵的條件，並討論其與 Iverson (2006) 於費區納表徵下所得之結果的關聯。本研究成果已寫成論文 ”Conditions under which affine representations are also Fechnerian under the power law of similarity on the Weber sensitivities”，且即將發表於 Journal of Mathematical Psychology 期刊（見附文）。

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Abstract

In this project we use the Fechner method (of taking the limit of sequence of integrals of jnd's) to construct the scales in a (weakly balanced) affine representation. We postulate a psychophysical law of similarity, and study its impact on the possible functional forms in the affine representation. Under this framework we further study the conditions under which affine representations are also Fechnerian, and link the results to the solutions in Iverson (2006) that was worked out within the Fechnerian framework. The work of this project has been summarized in a paper titled "*Conditions under which affine representations are also Fechnerian under the power law of similarity on the Weber sensitivities*" to be published in the *Journal of Mathematical Psychology* (see the attached file).

Keywords: affine representation; balance condition; Fechner method; Fechnerian representation; law of similarity

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Conditions under which affine representations are also Fechnerian under the power law of similarity on the Weber sensitivities

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We dedicate this paper to the memory of Duncan Luce whose vast body of work was instrumental in defining the field of mathematical psychology

Keywords:
Affine representation
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ABSTRACT

In a recent paper, Hsu, Iverson, and Doble (2010) examined some properties of a (weakly balanced) affine representation for choices, $\Psi(x, y) = F\left(\frac{u(x)-u(y)}{\sigma(y)}\right)$, and showed that using the Fechner method of integrating jnds, one can reconstruct the scales u and σ from the behavior of (Weber) sensitivities $\xi_s(x) = x + \Delta_s(x)$ (where $s = F^{-1}(\pi)$ and Δ_s is the jnd) in a neighborhood of $s = 0$. Following Iverson (2006b), in this article we impose a power law of similarity on the sensitivities, $\xi_s(\lambda x) = \lambda^{\iota(s)}\xi_{\eta(\lambda,s)}(x)$, and study its impact on u and σ in the affine representation. Especially, we specify the conditions for the first- and second-order derivatives of $\xi_s(x)$ with respect to s (and evaluated at $s \to 0$) under which the affine representation degenerates to a Fechnerian one. We also link the results to the solutions in Iverson (2006b), which was worked out within the Fechnerian framework.

1. Introduction

Let $(x, y) \in I \times I$ for some interval I of positive real numbers. Consider a system of choices $\Psi(x, y)$ which is a continuous mapping from $I \times I$ onto a positive interval. Such a system is called *regular* if (i) for each $y \in I$, $x \mapsto \Psi(x, y)$ is strictly increasing from I onto an interval $I_y \subset \mathbb{R}_+$, and (ii) for each $x \in I$, $y \mapsto \Psi(x, y)$ is strictly decreasing from I onto an interval $I'_x \subset \mathbb{R}_+$.[1] We define $q_\pi(y)$ via $x = q_\pi(y)$ iff $\Psi(x, y) = \pi$. In psychophysics, a standard *Fechnerian representation* for Ψ has been proposed (Falmagne, 1985; Luce & Galanter, 1963):

$$\Psi(x, y) = F\,(u(x) - u(y))\,, \tag{1}$$

in which the scales u and F are continuous and strictly increasing on a common interval of the positive reals.

Generalizing (1), Hsu et al. (2010) introduced an *affine representation*

$$\Psi(x, y) = F\,(u(x)h(y) + g(y))\,, \tag{2}$$

in which F is continuous, strictly increasing and differentiable with positive derivative, and h is assumed to vanish nowhere on I. The authors showed that within the framework of (2), a regular system of choices can always be *weakly balanced* (which, within the present context, is defined as $F^{-1}(\Psi(x, x)) = 0$ for all x in I):

$$\Psi(x, y) = F\left(\frac{u(x) - u(y)}{\sigma(y)}\right), \tag{3}$$

in which u is strictly increasing and σ is positive, and both are continuous and differentiable on I.

It is convenient to adopt $s = F^{-1}(\pi)$ in (3) as the index of the response criterion. Let $\Delta_s(x)$ be the *Weber function* representing the increment in stimulus x that is required to produce the s-indexed "just noticeable difference" (jnd), and write $\xi_s(x) = x + \Delta_s(x)$, called *(Weber) sensitivities*, for $q_{F(s)}(x)$. Thus $\xi_s(x)$ is defined for all x in an interval D_s and all s in the range of $F^{-1}(\pi)$. It follows that $\xi_s(x)$ is strictly increasing and continuous in both x and s. Hsu et al. (2010, Theorem 12 on p. 263) showed that using the Fechner method[2] of integrating jnds, $\lim_{s\to 0} \frac{\Delta_s(x)}{s\sigma(x)} = \frac{1}{u'(x)}$ (Falmagne, 1985; Iverson, 2006a; Krantz, 1971; Pfanzagl, 1962),

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[1] Note that the intervals I_y and I'_x are different. See footnote 1 in Hsu, Iverson, and Doble (2010) for explanation.

[2] In the literature Fechner used what he called the *Mathematical Auxiliary Principle* for constructing the sensory scale. His approach, however, was soundly criticized by Luce and Edwards (1958) as being flawed. Krantz (1971) pointed

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one can reconstruct the scales u and σ in (3) from the behavior of $\xi_s(x)$ in a neighborhood of $s = 0$. Specifically,

$$\sigma(x) = \sigma(x_0)\exp\int_{x_0}^{x}\frac{\nu(t)}{f(t)}\,dt \tag{4}$$

for any x_0 in I, and

$$u(x) = u(x_0) + \int_{x_0}^{x}\frac{\sigma(t)}{f(t)}\,dt, \tag{5}$$

where $f(x) = \left.\frac{\partial \xi_s(x)}{\partial s}\right|_{s=0}$, $\omega(x) = \left.\frac{\partial^2 \xi_s(x)}{\partial s^2}\right|_{s=0}$, and $\nu(x) = f'(x) - \frac{\omega(x)}{f(x)}$. In Section 4 we restate the theorem by providing an alternative formulation and proof (see Theorem 3).

The functions f and ω in (4) and (5) have an appealing interpretation in psychophysics. Consider discrimination probabilities from tasks consisting of a set of referents and comparison stimuli. Given a referent, the value of f is the derivative of the *inverse* of the psychometric function evaluated at $s = 0$. And f is a function of the referent (measured in ratio scale units). The second-order derivative ω can be interpreted in a similar way.

While (4) and (5) provide a characterization of the scales u and σ in (3) via the sensitivities ξ_s, in practice the formulas only lead to an approximation because the functional forms of f and ω are unknown. Imposing some empirically-based regularity properties on (3) can help narrow down the possible forms. An appealing candidate for regularity is the notion of "similarity".

In this article we impose a power law of similarity on the sensitivities and study its impact on u and σ in a weakly balanced affine representation. We start by introducing the concept of a law of similarity with a few examples. That leads to the presentation of a power law of similarity which is more closely related to the cases studied in Iverson (2006b). In Section 4 we review the Fechner method for reconstructing σ and μ in (3) and provide an alternative formulation and proof. We devote Section 5 to specifying the conditions for the first- and second-order derivatives of $\xi_s(x)$ with respect to s (and evaluated at $s \to 0$) under which the affine representation degenerates to a Fechnerian one. Last but not least, we link the results to the solutions in Iverson (2006b), which was worked out within the Fechnerian framework. We also briefly discuss the issue of characterizing the scales in (3) from a slightly different angle.

2. Similarity: a homogeneity law

We define *similarity* as follows. Let $\mathcal{F} = \{f_j(x)\,|\,j = 1, 2, \ldots, J\}$ be a finite collection of real-valued functions defined on the positive reals. We say that the collection $\mathcal{F}$ is *similar* if there are positive constants α_j, β_j and a function g such that $f_j(x) = \alpha_j g(\beta_j x)$. It follows that all members of $\mathcal{F}$ can be expressed in terms of any single member, say, f_k. Namely, $f_j(x) = \frac{\alpha_j}{\alpha_k} f_k(\frac{\beta_j}{\beta_k}x)$, which can be re-written as $f_j(\frac{\beta_k}{\beta_j}x) = \frac{\alpha_j}{\alpha_k} f_k(x)$.

In the following we give three examples of similarity across different contexts from one of the authors' previous work.

The first example concerns the photoreceptor sensitivities (as a function of log wavelength) in vision study. Observing that different classes of cone absorption spectra $S_k(\lambda)$, k = L, M, S receptors appear to have a common shape and differ only in the values of peak wavelengths λ_k, Iverson and Chubb (2013) related those sensitivities via a similarity form: $S_k(\lambda) = S_{k'}(\lambda(\lambda_{k'}/\lambda_k))$ for any indices $k, k' \in \{\mathrm{L}, \mathrm{M}, \mathrm{S}\}$.

The second example concerns partial masking in psychoacoustics. Pavel and Iverson (1981) used a matching function $\phi_n(x)$ to characterize the perceived loudness of a tone in quiet that matches a tone of intensity x embedded in a noise background of intensity n. They found that empirical data suggest a homogeneous relation of "shift invariance" for the family of matching functions: $\phi_{\lambda^\theta n}(\lambda x) = \lambda\phi_n(x)$ for any $\lambda > 0$ and some $0 < \theta < 1$ (see also Iverson & Pavel, 1981).

The third example, the focus of the present study, concerns the near-miss to Weber's law phenomenon (Doble, Falmagne, & Berg, 2003; Falmagne, 1985, 1994; Hsu et al., 2010). To have an insight into the regularity shown in this kind of data, Iverson (2006b) introduced a *law of similarity* on the sensitivities within the weakly balanced system: for all λ such that $\lambda x \in I$,

$$\xi_s(\lambda x) = \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(x), \tag{6}$$

in which $\gamma(\lambda, s)$ and $\eta(\lambda, s)$ are parameters which are themselves continuous functions in the two variables. Within this context and also considering the "shift invariance" example, we assume that $\gamma(\lambda, s)$ varies monotonically with λ and $\eta(\lambda, s)$ varies monotonically with s.

Note that setting $\lambda = 1$ in (6) yields $\xi_s(x) = \gamma(1, s)\xi_{\eta(1,s)}(x)$, implying that $\gamma(1, s) = 1$ if and only if $\eta(1, s) = s$. Also, employing the weakly balanced condition yields $\xi_0(\lambda x) = \gamma(\lambda, 0)\xi_{\eta(\lambda,0)}(x) = \lambda x$, implying that $\gamma(\lambda, 0) = \lambda$ if and only if $\eta(\lambda, 0) = 0$. We impose these "boundary conditions" (i.e., $\gamma(1, s) = 1$, $\eta(1, s) = s$, $\gamma(\lambda, 0) = \lambda$, and $\eta(\lambda, 0) = 0$) on (6), as they are reasonable assumptions in view of the last two examples. In an earlier study Iverson (2006b) invoked these boundary conditions and solved (6) for all families of sensitivities that admit a Fechnerian representation (1).

To further tighten up (6), we also assume that η is "multiplicatively translational" (Aczél, 1987) in the sense that $\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s))$ for all λ, λ' such that $\lambda x, \lambda' x, \lambda\lambda' x \in I$. This assumption puts some constraints on η. In particular, if $\eta(\lambda, s)$ is non-constant in λ, then fixing $s = s_0 \neq 0$ in the equation yields $\eta(\lambda\lambda', s_0) = \eta(\lambda', \eta(\lambda, s_0))$. Now write $\eta(\lambda, s_0) = t = H(\lambda)$. It follows from Aczél (1966, 1987) that H must be strictly monotonic, and one gets $\eta(\lambda', t) = \eta(\lambda\lambda', s_0) = H(\lambda\lambda') = H(\lambda' H^{-1}(t))$. i.e., $\eta(\lambda, s) = H(\lambda H^{-1}(s))$. Note that $\eta(\lambda, 0) = 0$ forces $H(0) = 0$.

3. A (power) law of similarity on the Weber sensitivities

Despite their innocuous appearances, η and γ in (6) are somewhat related. To see this, we iterate (6) to obtain

$$\xi_s(\lambda\lambda' x) = \gamma(\lambda\lambda', s)\xi_{\eta(\lambda\lambda',s)}(x) \tag{7}$$

$$= \gamma(\lambda, s)\xi_{\eta(\lambda,s)}(\lambda' x) = \gamma(\lambda, s)\gamma(\lambda', \eta(\lambda, s))\xi_{\eta(\lambda',\eta(\lambda,s))}(x). \tag{8}$$

We first note that if $\eta(\lambda, s)$ does not vary with λ, then the boundary condition $\eta(1, s) = s$ forces $\eta(\lambda, s) = s$ for all λ. Inserting this into (8) yields $\gamma(\lambda, s) = \lambda^{\iota(s)}$ for some function $\iota(s)$. Further, one sees from (7) and (8) that

$$\gamma(\lambda\lambda', s) = \gamma(\lambda, s)\gamma(\lambda', \eta(\lambda, s)) \tag{9}$$

if and only if

$$\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s)). \tag{10}$$

Eq. (10) is exactly the multiplicative-translation postulate mentioned above, so we focus on (9). If $\eta(\lambda, s)$ is non-constant in λ, then

out that the auxiliary principle that Fechner used can be reformulated in such a way that the sensory scale is constructed as the limit of a sequence of integrals (see also Pfanzagl, 1962). Specifically, the Fechnerian representation (1) assumes a family of functional equations of Abel's type: $u(\xi_s(x)) - u(x) = s$. The Krantz–Pfanzagl theorem states that one can construct the scale u via $u(x) - u(y) = \lim_{s\to 0+}\int_y^x \frac{s}{\Delta_s(z)}\,dz$. We call this the *Fechner method* and apply it to the affine representation.

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continuing our reasoning earlier, we fix (the same) $s = s_0 \neq 0$ in (9), yielding $\gamma(\lambda\lambda', s_0) = \gamma(\lambda, s_0)\gamma(\lambda', \eta(\lambda, s_0))$, which, with $K(\lambda) = \gamma(\lambda, s_0)$, can be rewritten as $K(\lambda\lambda') = K(\lambda)\gamma(\lambda', t)$. (Recall that $\eta(\lambda, s_0) = t = H(\lambda)$.) Thus $\gamma(\lambda', t) = \frac{K(\lambda\lambda')}{K(\lambda)} = \frac{K(H^{-1}(t)\lambda')}{K(H^{-1}(t))} = \frac{K\circ H^{-1}(H(\lambda' H^{-1}(t)))}{K\circ H^{-1}(t)}$. i.e., with $G = K \circ H^{-1}$, one obtains $\gamma(\lambda, s) = \frac{G(\eta(\lambda,s))}{G(s)}$. We summarize the result below.

Theorem 1. *Consider the law of similarity (6) in which η is multiplicatively translational: $\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s))$. Then either $\eta(\lambda, s) = s$ for all λ and $\gamma(\lambda, s) = \lambda^{\iota(s)}$, where $\iota(s)$ is a function of s, or there exist continuous and strictly monotonic functions H and G satisfying $H(0) = 0$ such that $\eta(\lambda, s) = H(\lambda H^{-1}(s))$ and $\gamma(\lambda, s) = \frac{G(\eta(\lambda,s))}{G(s)}$ (or equivalently, $\tilde{\gamma}(\lambda, t) = \frac{K(\lambda t)}{K(t)}$, where $K(\lambda) = \gamma(\lambda, s_0)$ and $t = H^{-1}(s)$).*

The form of $\tilde{\gamma}$ in Theorem 1 prompts an attempt to further characterize its behavior. A useful notion is *regular variation*: $\lim_{t\to 0} \tilde{\gamma}(\lambda, t) = \lim_{t\to 0} \frac{K(\lambda t)}{K(t)} = \lim_{t\to 0} \frac{\gamma(\lambda t, s_0)}{\gamma(t, s_0)} = \lambda^{\delta_{s_0}}$ for some real number δ_{s_0}. Adhering to this property, we consider a situation wherein the multiplier γ in (6) takes a power-function form globally (as in the first subcase). Such a presumption is supported in some important applications. An example, though from a different context outside the weakly balanced system, is the homogeneous relation of shift invariance for the family of matching functions mentioned earlier (Iverson & Pavel, 1981; Pavel & Iverson, 1981).

Another example, the focus of this study, involves a power-function formulation for the sensitivities. Iverson (2006b) discussed two such instances of law of similarity that capture the near-miss to Weber's law phenomenon well: one is the power function advocated by Falmagne (dubbed "Falmagne's power law")

$$\xi_s(\lambda x) = \lambda^{\iota(s)} \xi_s(x) \tag{11}$$

and the other (dubbed "psychophysical power law") takes the form

$$\xi_s(\lambda x) = \lambda \xi_{\eta(\lambda,s)}(x). \tag{12}$$

Both are special cases of Theorem 1.

Taking the two instances (11) and (12) into account, in this article we focus on a slightly restrictive form of (6) by assuming that the multiplier is a power function: $\gamma(\lambda, s) = \lambda^{\iota(s)}$, in which $\iota(s)$ is positive, continuous, and twice differentiable in s, with $\iota(0) = 1$. This (η-*multiplicatively translational*) *power law of similarity* reads

$$\xi_s(\lambda x) = \lambda^{\iota(s)} \xi_{\eta(\lambda,s)}(x) \tag{13}$$

for all λ such that $\lambda x \in I$. The parameter ι in (13) is constrained if $\eta(\lambda, s)$ varies with λ, under which (12) comes about as a logical consequence. To see this, we iterate (13) and obtain $\iota(s) = \iota(\eta(\lambda, s))$ if and only if $\eta(\lambda\lambda', s) = \eta(\lambda', \eta(\lambda, s))$. If $\eta(\lambda, s)$ varies with λ, then $\iota(s)$ must be a constant function of s. In such a case since the system is weakly balanced, we have $\lambda = \xi_0(\xi_0(\lambda)) = \xi_0(\lambda^{\iota(0)}\xi_0(1)) = \lambda^{\iota(0)^2}\xi_0(\xi_0(1)) = \lambda^{\iota(0)^2}$, indicating that $\iota(s) = 1$ for all s. Furthermore, since $\eta(\lambda, s) = H(\lambda H^{-1}(s))$ (see Theorem 1), one can define $\varrho = H^{-1}(s)$ and $\tilde{\xi}_\varrho(x) = \xi_s(x)$, and (13) is simplified to $\tilde{\xi}_\varrho(\lambda x) = \lambda \tilde{\xi}_{\lambda\varrho}(x)$. We have the following result.

Corollary 2. *Consider the (η-multiplicatively translational) power law of similarity (13) in a weakly balanced system. If $\eta(\lambda, s)$ varies with λ, then $\iota(s) = 1$ for all s. In such a case with $\varrho = H^{-1}(s)$ and $\tilde{\xi}_\varrho(x) = \xi_s(x)$, (13) is simplified to $\tilde{\xi}_\varrho(\lambda x) = \lambda \tilde{\xi}_{\lambda\varrho}(x)$.*

Theorem 1 and Corollary 2 are useful in confining the scales in the affine representation (see the proof of Theorem 5 in Section 5).

4. Integrating jnds to recover weakly balanced affine representations: an alternative proof

The following theorem is a restatement (though using a slightly different formulation) of Theorem 12 in Hsu et al. (2010). The proof presented here is slightly different from and shorter than the proof in Hsu et al. (2010).

Theorem 3. *Suppose that a regular system of choices can be represented in the form of a weakly balanced affine representation (3), in which the scales $u(x)$ and $\sigma(x)$ are continuously differentiable on I. Also, suppose that the sensitivities $\xi_s(x)$ are, for each x belonging to I, twice differentiable in s in a neighborhood of $s = 0$. Then*

$$u(x) = u(x_0) + u'(x_0) \int_{x_0}^{x} \left[\exp\left(-\int_{x_0}^{y} \frac{\omega(t)}{f^2(t)}\, dt \right) \right] dy \tag{14}$$

for any x_0 in I, and

$$\sigma(x) = u'(x_0) f(x) \exp\left[-\int_{x_0}^{x} \frac{\omega(t)}{f^2(t)}\, dt \right], \tag{15}$$

where $f(x) = \left.\frac{\partial \xi_s(x)}{\partial s}\right|_{s=0}$ and $\omega(x) = \left.\frac{\partial^2 \xi_s(x)}{\partial s^2}\right|_{s=0}$.

Proof. The equation for sensitivities from (3) reads

$$u(\xi_s(x)) = u(x) + s\sigma(x).$$

Differentiating the above equation with respect to the criterion variable s, we get

$$\frac{\partial \xi_s(x)}{\partial s} u'(\xi_s(x)) = \sigma(x). \tag{16}$$

Differentiate again with respect to s to give

$$\left(\frac{\partial \xi_s(x)}{\partial s}\right)^2 u''(\xi_s(x)) + \frac{\partial^2 \xi_s(x)}{\partial s^2} u'(\xi_s(x)) = 0. \tag{17}$$

Take the limit $s \to 0$ in each of (16) and (17), and recall that within a weakly balanced system, $\xi_s(x) \to x$ as $s \to 0$. We obtain, with $f(x) = \left.\frac{\partial \xi_s(x)}{\partial s}\right|_{s=0}$ and $\omega(x) = \left.\frac{\partial^2 \xi_s(x)}{\partial s^2}\right|_{s=0}$,

$$f(x)u'(x) = \sigma(x) \tag{18}$$

and

$$f^2(x)u''(x) + \omega(x)u'(x) = 0. \tag{19}$$

We then integrate (19) to recover $u(x)$; the scale $\sigma(x)$ then can be recovered from (18). Specifically, from (19) we obtain

$$\frac{u''(x)}{u'(x)} = \frac{d \ln u'(x)}{dx} = -\frac{\omega(x)}{f^2(x)},$$

from which a first integration gives

$$u'(x) = u'(x_0) \exp\left[-\int_{x_0}^{x} \frac{\omega(t)}{f^2(t)}\, dt \right] \tag{20}$$

for any x_0 in I, and a second integration gives

$$\frac{u(x) - u(x_0)}{u'(x_0)} = \int_{x_0}^{x} \left[\exp\left(-\int_{x_0}^{y} \frac{\omega(t)}{f^2(t)}\, dt \right) \right] dy,$$

leading to (14). The form of σ (15) can be obtained from (18) and (20). □

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It is straightforward to verify that (15) and (4) are identical, and so are (14) and (5). Note that differentiating (15) and using (20) yields $\sigma'(x) = u'(x)\nu(x)$, in which $\nu(x) = f'(x) - \frac{\omega(x)}{f(x)}$, an expression involving the first- and second-derivatives of $\xi_s(x)$ that can be used to characterize the relationship of affine and Fechnerian representations. Indeed, Hsu et al. (2010) proved a corollary[3] just for that, and used a special form (see (23)) of Falmagne's power law (Falmagne, 1985; Iverson, 2006b) to illustrate the application. We restate the corollary (with more details) below, as it is helpful for giving insight into the impact of the power law of similarity (13) on (3).

Corollary 4. *In Theorem 3, define $\nu(x) = f'(x) - \frac{\omega(x)}{f(x)}$. If $\nu(x) = A$, a constant, then either σ is constant (in which case $A = 0$) or $\sigma(x) = Au(x)+B$ ($A \neq 0$), and the weakly balanced affine representation (3) is equivalent to a Fechnerian representation (1). The converse is also true.*

5. Conditions under which an affine representation is Fechnerian

We study the impact of (13) on the scales u and σ in the weakly balanced affine representation (3) using the Fechner method (via the forms of f and ω in Theorem 3). The following theorem not only spells out the constraints of ι and η in (13) and f in (3), it also specifies the conditions for ω under which (3) reduces to (1).

Theorem 5. *In Theorem 3, suppose that the weakly balanced affine representation (3) also satisfies the (η-multiplicatively translational) power law of similarity (13). Then one of the following three cases holds:*

(I) *$\iota'(0) = 0$, $\eta(\lambda, s) = s$, and $f(x) = Cx$ for some constant $C > 0$. Moreover, (3) reduces to (1) if $\omega(x) = Ax$ for some constant A.*
(II) *$\iota'(0) \neq 0$, $\eta(\lambda, s) = s$, and $f(x) = \iota'(0)x\ln x + Bx$ for some constant B. Moreover, (3) reduces to (1) if $\omega(x) = (\iota'(0))^2 x(\ln x)^2 + (2B\iota'(0) + \iota''(0))x\ln x + (B^2 + \frac{B\iota''(0)}{\iota'(0)})x$.*
(III) *$\iota(s) = 1$ for all s, $\eta(\lambda, s) = \lambda^B s$, and $f(x) = Ax^{B+1}$ for some constants $A > 0$ and $B \neq 0$. Moreover, (3) reduces to (1) if $\omega(x) = A^2(B+1)x^{2B+1}$.*

Proof. We differentiate ξ in (13) with respect to s:

$$\frac{\partial \xi_s(\lambda x)}{\partial s} = \frac{\partial}{\partial s}\left(\lambda^{\iota(s)}\xi_{\eta(\lambda,s)}(x)\right) = \lambda^{\iota(s)}(\ln\lambda)\iota'(s)\xi_{\eta(\lambda,s)}(x) + \lambda^{\iota(s)}\frac{\partial \xi_\eta(x)}{\partial \eta}\frac{\partial \eta(\lambda,s)}{\partial s}. \quad (21)$$

Let $s \to 0$, then

$$\left.\frac{\partial \xi_s(\lambda x)}{\partial s}\right|_{s=0} = \lambda(\ln\lambda)\iota'(0)x + \lambda\left.\frac{\partial \xi_\eta(x)}{\partial \eta}\right|_{\eta=0}\left.\frac{\partial \eta(\lambda,s)}{\partial s}\right|_{s=0}.$$

Define

$$\phi(x) = \left.\frac{\partial \xi_s(x)}{\partial s}\right|_{s=0}\Big/ x, \qquad k(\lambda) = \left.\frac{\partial \eta(\lambda,s)}{\partial s}\right|_{s=0},$$
$$\ell(\lambda) = \iota'(0)\ln\lambda,$$

and the above equation can be rewritten as

$$\phi(\lambda x) = \ell(\lambda) + \phi(x)k(\lambda),$$

a (generalized) Pexider functional equation. We have three families of continuous solutions for ϕ (Aczél, 1966), subject to the requirement that ϕ is positive. Since $\left.\frac{\partial \xi_s(x)}{\partial s}\right|_{s=0} = f(x) = x\phi(x)$, we know $f(x)$ in each case.

(I) $\phi(x) = C > 0$, $k(\lambda) = 1 - \frac{\iota'(0)}{C}\ln\lambda$, and so $f(x) = Cx$.
(II) $\ell \neq 0$ (and so $\iota'(0) \neq 0$), $\phi(x) = \iota'(0)\ln x + B > 0$ for some constant B, and $k = 1$. Thus $f(x) = \iota'(0)x\ln x + Bx$.
(III) $\ell = 0$ (and so $\iota'(0) = 0$), $k(x) = x^B$, and $\phi(x) = Ax^B > 0$ so that $f(x) = Ax^{B+1}$ for some constants $A > 0$ and $B \neq 0$.

It remains to find the functional form for ω, since together ω and f determine u, σ (see (14), (15) in Theorem 3), and ν (see Corollary 4). Now $\omega(x) = \left.\frac{\partial^2 \xi_s(x)}{\partial s^2}\right|_{s=0}$, so we compute a second derivative of (21):

$$\frac{\partial^2 \xi_s(\lambda x)}{\partial s^2} = \ln\lambda\left((\iota'(s))^2\lambda^{\iota(s)}(\ln\lambda) + \lambda^{\iota(s)}\iota''(s)\right)\xi_{\eta(\lambda,s)}(x) + 2\lambda^{\iota(s)}(\ln\lambda)\iota'(s)\frac{\partial \xi_\eta(x)}{\partial \eta}\frac{\partial \eta(\lambda,s)}{\partial s} + \lambda^{\iota(s)}\left[\frac{\partial^2 \xi_\eta(x)}{\partial \eta^2}\left(\frac{\partial \eta(\lambda,s)}{\partial s}\right)^2 + \frac{\partial \xi_\eta(x)}{\partial \eta}\frac{\partial^2 \eta(\lambda,s)}{\partial s^2}\right].$$

Take the limit as $s \to 0$ in the above equation and define

$$\zeta(x) = \left.\frac{\partial^2 \xi_s(x)}{\partial s^2}\right|_{s=0}\Big/ x, \qquad z(\lambda) = \left.\frac{\partial^2 \eta(\lambda,s)}{\partial s^2}\right|_{s=0},$$
$$\tau(\lambda) = [(\iota'(0))^2\ln\lambda + \iota''(0)]\ln\lambda.$$

We obtain

$$\zeta(\lambda x) = \tau(\lambda) + 2\ell(\lambda)\phi(x)k(\lambda) + [\zeta(x)k^2(\lambda) + \phi(x)z(\lambda)]. \quad (22)$$

We now provide the solutions to each of the three cases.

For case (I), the term $\iota'(0)$ in $k(\lambda) = \left.\frac{\partial \eta(\lambda,s)}{\partial s}\right|_{s=0} = 1 - \frac{\iota'(0)}{C}\ln\lambda$ must equal 0, otherwise we have a situation in which $\iota(s)$ is not constant and $\left.\frac{\partial \eta(\lambda,s)}{\partial s}\right|_{s=0}$ varies with λ—contradictory to Corollary 2. (i.e., by Theorem 1 and Corollary 2 we must have $\eta(\lambda, s) = s$ and so $k(\lambda) = 1$, $\ell(\lambda) = 0$, and $z(\lambda) = 0$ for all λ.) Inserting these values into (22) gives

$$\zeta(\lambda x) = \tau(\lambda) + \zeta(x),$$

a functional equation with known continuous solutions.

Subcase (I1) $\zeta(x) = A$ and so $\omega(x) = Ax$. By Corollary 4 we have $\nu(x) = \frac{C^2-A}{C}$, a constant, indicating that the weakly balanced affine representation degenerates to a Fechnerian one. Moreover, $\tau(\lambda) = 0$ and so $\iota''(0) = 0$.

Subcase (I2) $\zeta(x) = a\ln x + b$ ($a \neq 0$) and so $\omega(x) = ax\ln x + bx$. We have $\nu(x) = \frac{C^2-b}{C} - \frac{a}{C}\ln x$, indicating that the affine representation does *not* degenerate to a Fechnerian one. In this case $\iota''(0) \neq 0$.

For case (II), $\iota'(0) \neq 0$ implies that $\iota(s)$ is not constant. Theorem 1 and Corollary 2 apply and we have $\eta(\lambda, s) = s$ and so $z(\lambda) = 0$ for all λ. Also recall that $k = 1$. Inserting these values into (22) and then differentiating it with respect to x gives

$$\zeta'(\lambda x)\lambda = \phi'(x)2\ell(\lambda) + \zeta'(x).$$

Recall that $\phi(x) = \iota'(0)\ln x + B$ and so $\phi'(x) = \iota'(0)/x$. Multiplying both sides of the above equation by x, we obtain, with $R(x) = x\zeta'(x)$ and $V(\lambda) = 2\iota'(0)\ell(\lambda) = 2(\iota'(0))^2\ln\lambda$, a Pexider functional equation

$$R(\lambda x) = V(\lambda) + R(x).$$

[3] The proof relies on an established theorem stating that (3) reduces to (1) if and only if σ is constant or σ is not constant but $u = A\sigma + B$ for some constants A and B. See also Theorem 6 in Hsu et al. (2010) for a slightly more general statement.

Thus $R(x) = 2(\iota'(0))^2 \ln x + b$ for some constant b and so the form of ζ is obtainable, which can be inserted back to (22) to yield $b = 2B\iota'(0) + \iota''(0)$. We have $\omega(x) = (\iota'(0))^2 x(\ln x)^2 + (2B\iota'(0) + \iota''(0))x\ln x + cx$, where c is a constant. The parameter c is constrained for the Fechnerian case. A little algebra shows that ν in Corollary 4 is a constant if $c = B^2 + \frac{B\iota''(0)}{\iota'(0)}$.

For case (III), $k(\lambda) = \lambda^B$ being non-constant imposes a strong constraint on the functional form of η. By Corollary 2, we have $\iota(s) = 1$ for all s and, with $H(\lambda) = \lambda^B$, $\eta(\lambda, s) = H(\lambda H^{-1}(s)) = \lambda^B s$. Inserting $\tau = 0$, $\ell = 0$, $z = 0$ and $k(\lambda) = \lambda^B$ into (22) gives

$$\zeta(\lambda x) = \zeta(x)\lambda^{2B}$$

with two continuous solutions.

Subcase (III1) $\zeta(x) = 0$ and so $\omega(x) = 0$ for all x. We have $\nu(x) = A(B+1)x^B$, and thus ν is constant if and only if $B = -1$. i.e., the representation is Fechnerian if $f(x) = A$ for all x in I.

Subcase (III2) $\zeta(x) = ax^{2B}$ $(a \neq 0)$ and so $\omega(x) = ax^{2B+1}$. Applying Corollary 4 gives $\nu(x) = A(B+1)x^B - \frac{a}{A}x^B$. Thus the representation is Fechnerian if and only if $A(B+1) = \frac{a}{A}$. In this case $B \neq -1$.

Note that by Corollary 4, for both subcases of (III) the representation is Fechnerian if and only if σ is constant. □

Using the same method, one can show that, for (3) under the more general law of similarity (6), there are exactly three possible forms of f: (i) $f(x) = Cx$ $(C > 0)$, (ii) $f(x) = Ax\ln x + Bx$ $(A \neq 0)$, and (iii) $f(x) = Ax^{B+1} + Cx$ $(A, B \neq 0)$. As can be seen, they resemble (generically) the three forms of f in Theorem 5. Among them (i) is accompanied with three possible forms of ω, and (ii) and (iii) are each accompanied with two possible forms of ω.[4] It follows from Corollary 4 that the could-degenerate-to-Fechnerian scenario resembles closely the corresponding subcases in Theorem 5.[5]

We pause here to note that if ι in (13) is further assumed to be strictly monotonic, then subcase (I2) in the proof of Theorem 5 can be ruled out, meaning that (I) only admits the Fechnerian representation. Moreover, case (II) of Theorem 5 resembles Falmagne's power law (11), for which Hsu et al. (2010) have shown that it takes (actually, is equivalent to) a special form

$$\xi_s(x) = x^{\iota(s)} x_0^{1-\iota(s)}, \tag{23}$$

where $x_0 > x$ for all x is the maximum value at which all $\ln \frac{\xi_s(x)}{x}$ curves intersect on the $\ln x$ axis.[6] The authors proved that (23) forces the weakly balanced affine representation to be a Fechnerian one (see Theorem 11 on p. 262). Indeed, by slightly strengthening the properties of the functions σ in (3) and ι in (11) (without explicitly spelling out the special form as in Theorem 11 of Hsu et al., 2010), one can show that Falmagne's power law (11) only admits the Fechnerian representation.

[4] For (i), they are $\omega(x) = ax$, $\omega(x) = ax\ln x + bx$ $(a \neq 0)$, and $\omega(x) = ax^{b+1} + cx$ $(a, b \neq 0)$. For (ii), they are $\omega(x) = cx\ln x + bx$ and $\omega(x) = \frac{a}{2}x(\ln x)^2 + bx\ln x + cx$ $(a \neq 0)$. For (iii), they are $\omega(x) = \frac{c}{B}x^{B+1} + bx$ and $\omega(x) = \frac{a}{B+b}x^{B+b+1} + \frac{c}{B}x^{B+1} + dx$ $(a, b \neq 0)$.

[5] For $f(x) = Ax^{B+1} + Cx$, there are two scenarios for the could-degenerate-to-Fechnerian subcase, depending on whether $C = 0$. The one described in Subcase (III) of Theorem 5 deals with the case in which $C = 0$. See our remark after Theorem 8.

[6] The reason is as follows. Fixing any $x_{1,s} \in I$ in (11) and defining $\lambda x_{1,s} = y$, one gets $\xi_s(y) = (\frac{y}{x_{1,s}})^{\iota(s)}\xi_s(x_{1,s}) = y^{\iota(s)}x_{1,s}^{-\iota(s)}\xi_s(x_{1,s})$. Thus (11) can be re-written as $\xi_s(x) = x^{\iota(s)}K(s)$, where $K(s) = x_{1,s}^{-\iota(s)}\xi_s(x_{1,s})$. The graphs of $\ln \frac{\xi_s(x)}{x}$ versus $\ln x$ are linear with slopes $\iota(s) - 1$. In particular, $\ln \frac{\xi_0(x)}{x} = 0$ for all $x \in I$. If $\iota(s)$ varies with s, then those lines intersect at the coordinates $(\ln x_0, 0)$ (where $x_0 > x$ for all x) and so $K(s) = x_0^{1-\iota(s)}$.

Theorem 6. *Suppose that Falmagne's power law (11) holds for a weakly balanced affine representation (3). Also suppose that the functions ι in (11) and σ in (3) are continuous, strictly monotonic and never zero. Then the representation is Fechnerian.*

Proof. We first apply (3) to $\xi_s(\lambda x)$ to get

$$\xi_s(\lambda x) = u^{-1}(u(\lambda x) + s\sigma(\lambda x)).$$

Now fix $x = x_0$ and define a new variable $y = \lambda x_0$. Applying (11) to the above equation and taking logarithms on both sides gives

$$\ln \circ u^{-1}(u(y) + s\sigma(y)) = \iota(s)\ln\left(\frac{y}{x_0}\right) + \ln \xi_s(x_0),$$

which, with $g(y) = \ln(\frac{y}{x_0}) = t$, $\tilde{u} = u \circ g^{-1}$ and $\tilde{\sigma} = \sigma \circ g^{-1}$, can be rewritten as

$$\ln \circ u^{-1}\left(\tilde{u}(t) + s\tilde{\sigma}(t)\right) = t\iota(s) + \ln \xi_s(x_0), \tag{24}$$

where $\tilde{\sigma}$ and ι are continuous, strictly monotonic and never zero. The solution to (24), due to Falmagne and Lundberg (2000) (see Theorem 1.1 on page 203), ensures that $u(y) = u \circ g^{-1}(t) = \tilde{u}(t)$ and $\sigma(y) = \sigma \circ g^{-1}(t) = \tilde{\sigma}(t)$ are linearly related, and the result follows immediately (cf. footnote 3). □

In Theorem 5, the corresponding scales of u and σ in the three cases, for which the weakly balanced affine representation (3) degenerates to a Fechnerian representation (1), are also readily accessible by inserting the forms of f and ω into (14) and (15). Straightforward calculations yield the following result.

Corollary 7. *In Theorem 5, the corresponding scales of u and σ, for which the weakly balanced affine representation (3) degenerates to a Fechnerian representation (1), are given below.*[7]

(I) *if $A = C^2$, then σ is constant and u takes a logarithmic form. If $A \neq C^2$, then*

$$\sigma(x) = \sigma(x_1)\left(\frac{x}{x_1}\right)^{1-\frac{A}{C^2}}$$

and

$$u(x) = u(x_1) + \frac{\sigma(x_1)}{C\left(1 - \frac{A}{C^2}\right)}\left(\left(\frac{x}{x_1}\right)^{1-\frac{A}{C^2}} - 1\right)$$

for any x_1 in I.

(II) *If $\iota'(0) = \frac{\iota''(0)}{\iota'(0)}$, then σ is constant and u takes a log–log form. If $\iota'(0) \neq \frac{\iota''(0)}{\iota'(0)}$, then*

$$\sigma(x) = \sigma(x_1)\left(\frac{\iota'(0)\ln x + B}{\iota'(0)\ln x_1 + B}\right)^{1-\frac{\iota''(0)}{(\iota'(0))^2}}$$

and

$$u(x) = u(x_1) + \frac{\sigma(x_1)}{\iota'(0) - \frac{\iota''(0)}{\iota'(0)}}\left(\left(\frac{\iota'(0)\ln x + B}{\iota'(0)\ln x_1 + B}\right)^{1-\frac{\iota''(0)}{(\iota'(0))^2}} - 1\right)$$

for any x_1 in I.

(III) *σ is constant and u is a power function.*

[7] The constants A and C in (I) and B in (II) refer to the corresponding ones in Theorem 5.

Recall that (3) reduces to (1) if and only if σ is constant or σ is not constant but is linearly related to u (Hsu et al., 2010; Iverson & Pavel, 1981). It is clear that all three cases in Corollary 7 are consistent with this statement. Importantly, Corollary 7 indicates that u (and σ when it is not constant) take either a logarithmic, a log–log, or a power-function form.[8]

6. Examples and relations to the solutions in Iverson (2006b)

We mentioned earlier that an example for case (II) of Theorem 5 is the power law for the sensitivities ξ_s advocated by Falmagne and colleagues to account for the near-miss to Weber's law phenomenon (Doble et al., 2003; Doble, Falmagne, Berg, & Southworth, 2006; Falmagne, 1985). As discussed in Hsu et al. (2010) and rephrased earlier (see (23) and footnote 6), within the weakly balanced system of affine representations, Falmagne's power law (11) takes a special form $\xi_s(x) = x^{\iota(s)}x_0^{1-\iota(s)}$, where $x_0 > x$ for all $x \in I$. Straightforward differentiations of (23) with respect to s reveals special instances of f and w (as $s \to 0$) and, using the notation of Theorem 5, we have $B = -\iota'(0)\ln x_0$. Indeed, the result from Doble et al. (2006) indicates that $\iota'(0) < 0$ and thus $B > -\iota'(0)\ln x$ for all x in I. Further, under the Fechnerian framework (1), $\iota(s)$ in (23) is an exponential function of s (Falmagne, 1994; see also Eq. (11c) on p. 286 of Iverson, 2006b). These results complement the illustrative analysis described in Hsu et al. (2010).

For case (III) of Theorem 5, we also find an example from the study of Jesteadt, Wier, and Green (1977) of the near-miss to Weber's law phenomenon, for which the data pattern can be re-captured (to a good approximation) by a special form of (13) in which $\iota(s) = 1$ for all s (see also (12) and Corollary 2). As discussed in Iverson (2006b), under the Fechnerian framework (1), the homogeneity relation of sensitivities imposed by this model takes the form $\xi_s(x) = (x^\beta + \frac{s}{\alpha})^{1/\beta}$ $(\alpha, \beta > 0)$, with the scale u a power function $u(x) = \alpha x^\beta + \gamma$ (see Eq. (13b) on p. 286). This is consistent with case (III) of Theorem 5 by setting $B = -\beta$ and $A = \frac{1}{\alpha\beta}$.

Further inspection reveals that the simplest solution for u in Iverson (2006b, Eqs. (11a) and (13a) on pp. 285–286) also is compatible with case (I) of Theorem 5. Indeed, with $\iota(s) = 1$ for all s (and so $\iota'(0) = \iota''(0) = 0$) in case (I), we have Weber's law. In such a case $\xi_s(x) = e^{s/\alpha}x$ $(\alpha > 0)$.

However, *none* of the last six solutions for u in Iverson (2006b, Eqs. (15a)–(15f) on p. 287) appear in Corollary 7, suggesting that they are not derived under the power law of similarity (13), but rather are consequences of other forms of γ in (6). A further examination reveals that the multiplier γ that gives rise to these six solutions must satisfy $\frac{\partial\gamma(\lambda,s)}{\partial s}\Big|_{s=0} = a\lambda(1-\lambda^B)$ $(a, B \neq 0)$ (in addition to $\gamma(1, s) = 1$ and $\gamma(\lambda, 0) = \lambda$). For example, (15a) in Iverson (2006b) states that $\xi_s(x) = T(e^{s/\alpha}(x/T)^\beta + (e^{s/\alpha} - 1))^{1/\beta}$. If setting $T = 1$ (which can be achieved by scaling the variable x), then a computation shows that $\gamma(\lambda, s) = [e^{s/\alpha}(\lambda^\beta - 1) + 1]^{1/\beta}$, a form satisfying the above requirement. For the other five cases, the functional forms of γ are similar, except that the sign pattern differs in various places. A similar derivation of (15a) in Iverson (2006b) also gives $\eta(\lambda, s) = -\alpha\ln[\lambda^{-\beta}(e^{-s/\alpha} - 1) + 1]$, for which one can solve $\eta(\lambda, s) = H(\lambda H^{-1}(s))$, yielding $H(s) = -\alpha\ln(ks^{-\beta} + 1)$ for some constant $k \neq 0$. All these are special instances of Theorem 1. In fact, as mentioned after Theorem 5, within (3) and under the more general law of similarity (6), it can be shown that one set of conditions involves $f(x) = Ax^{B+1} + Cx$ for some constants A, $B \neq 0$ and C. In such a case for (3) to degenerate to (1), one must have $\frac{\partial\eta(\lambda,s)}{\partial s}\Big|_{s=0} = \lambda^B$. Obviously, both $\eta(\lambda, s) = \lambda^B s$ (case (III) of Theorem 5) and $\eta(\lambda, s) = -\alpha\ln[\lambda^B(e^{-s/\alpha} - 1) + 1]$ (we write $B = -\beta$) satisfy this condition.

7. Discussion and concluding remarks

In general, u and σ in (3) are not tidied up enough under the impact of (13) for further simplification. For instance, for subcase (I2) in the proof of Theorem 5 in which the affine representation does *not* degenerate to a Fechnerian one, since $\iota'(0) = 0$ and $\iota''(0) \neq 0$, the multiplier $\gamma(\lambda, s)$ in (13) is not monotonic in s. For this peculiar case the scales in (3) take a complicated form[9]:

$$u(x) = u(x_1) + Ke^{-\frac{1}{4}}\frac{\sqrt{\pi}}{2}\operatorname{erfi}\left(\ln x + \frac{1}{2}\right) + D,$$

where $K \neq 0$, $\operatorname{erfi}(x) = -i\operatorname{erf}(ix)$ and $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

To have some insight into this issue, we inspect the constraints imposed by the law of similarity (6) on the weakly balanced affine system (3) from a slightly different angle. As mentioned previously, using the Fechner method one can obtain three cases for ϕ and thus f: $f(x) = Cx$ $(C > 0)$, $f(x) = Ax\ln x + Bx$ $(A \neq 0)$, or $f(x) = Ax^{B+1} + Cx$ $(A, B \neq 0)$. Also recall that $f(x)u'(x) = \sigma(x)$ (see (18)). One then uses integration by parts (whenever applicable) to express u in terms of σ and σ'.

Theorem 8. *In Theorem 3, suppose that the weakly balanced system (3) also satisfies the law of similarity (6). Then u is given by one of the following forms.*

(I)

$$u(x) = \frac{1}{C}\left[\sigma(x)\ln x - \int\sigma'(x)\ln x\,dx\right] + D,$$

where $C > 0$, D are constants.

(II)

$$u(x) = \frac{1}{A}\left[\sigma(x)\ln\left(\ln x + \frac{B}{A}\right) - \int\sigma'(x)\ln\left(\ln x + \frac{B}{A}\right)dx\right] + D,$$

where $A \neq 0$ and B, D are constants.

(III) *If $C = 0$,*

$$u(x) = \frac{-1}{AB}\left[\sigma(x)x^{-B} - \int\sigma'(x)x^{-B}\,dx\right] + D,$$

where $A > 0$, $B \neq 0$ and D are constants.

If $C \neq 0$,

$$u(x) = \frac{1}{C}\left\{\sigma(x)\ln x - \int\sigma'(x)\ln x\,dx - \frac{1}{B}\left[\sigma(x)\ln\left(x^B + \frac{C}{A}\right) - \int\sigma'(x)\ln\left(x^B + \frac{C}{A}\right)dx\right]\right\} + D,$$

where A, $B \neq 0$ and D are constants.

It is clear from Theorem 8 that the cases in which σ is constant or σ and u are linearly related[10] reduce (3) to (1), for which all the

[8] Keep in mind that in Corollary 7 for the cases in which u and σ are linearly related, we have $F\left(\frac{u(x)-u(y)}{\sigma(y)}\right) = F\left(\frac{(A\sigma(x)+B)-(A\sigma(y)+B)}{\sigma(y)}\right) = F\left(A\left(\frac{\sigma(x)}{\sigma(y)} - 1\right)\right) = F^*(u^*(x) - u^*(y))$, meaning that the scale u in (3) needs to be transformed to $u^* = \ln\sigma$ when it is expressed in terms of (1).

[9] We simplify the matter by constraining some of the constants in the solution.

[10] In such a case we have $\frac{\sigma'(x)}{\sigma(x)} = g'(x)$, where $g(x)$ takes one of the forms: $\ln x$, $\ln(\ln x + \frac{B}{A})$, x^{-B}, or $(\ln x - \frac{1}{B}\ln(x^B + \frac{C}{A}))$. Note that a logarithmic transformation of scale is required for u (as mentioned in footnote 8).

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solutions of u and ξ were given in Iverson (2006b). Especially, the last six solutions for u in Iverson (2006b) are covered by (III) in which $C \neq 0$. However, for the non-degenerate (to Fechnerian) case it remains challenging to spell out the functional form for u (and σ) if no further constraints are imposed.[11]

In a broader sense, Eqs. (6) and (13) are special cases of the homogeneity relation

$$\Upsilon_s(\lambda x) = \gamma(\lambda, s)\Upsilon_{\eta(\lambda,s)}(x), \tag{25}$$

where the form Υ is empirically based, and good candidates for it (at least) in psychophysics include the sensitivity function ξ and the Weber function Δ. We mentioned previously that Iverson (2006b) solved (6) for all families of sensitivities that admit a Fechnerian representation (1). On the other hand, as pointed out by Falmagne (1985, 1994), the commonly adopted form $\Delta_s(\lambda x) = \lambda^{\beta(s)}\Delta_s(x)$ (which is a special case of the Weber function-based power law of similarity) with $\beta(s)$ varying with s is *incompatible* with the Fechnerian representation (see also Doble et al., 2003; Iverson, 2006b). In fact, taking Δ for Υ in (25) and considering only the case in which $\eta(\lambda, s) = s$ for all λ, it can be shown that $\Delta_s(x) \propto x$ for any given s and thus Weber's law is *implicitly* enforced. Formally,

Theorem 9. *For the Weber function-based law of similarity $\Delta_s(\lambda x) = \gamma(\lambda, s)\Delta_{\eta(\lambda,s)}(x)$, if $\eta(\lambda, s) = s$ for all λ, then (3) reduces to (1) if and only if $\gamma(\lambda, s) = \lambda$ for all s, i.e., Weber's law holds.*

Theorem 9 only covers the simplest case; when $\eta(\lambda, s)$ varies with λ the issue becomes complicated. An interesting follow-up study is to explore the intertwined relationship between the sensitivity function- and Weber function-based laws of similarity under (3).

Finally, we mention that the Fechner method of integrating jnds used in the present work can be applied to the more general case of affine representation,

$$\Psi(x, y) = F\left(\frac{u(x) - v(y)}{\sigma(y)}\right). \tag{26}$$

To see this, following Iverson (2006a) we redefine the sensitivities as $u(\xi_s(x)) - v(x) = s\sigma(x)$, so that $\xi_0(x) = u^{-1} \circ v(x)$ and the notion of jnds can be extended to $d_s(x) = \xi_s(x) - \xi_0(x)$. We then generalize the Krantz–Pfanzagl theorem to

$$u(x) - u(y) = \lim_{s\to 0+} \int_y^x \frac{s\sigma(z)}{d_s(\xi_0^{-1}(z))}\, dz.$$

Using similar arguments as in the proof of Theorem 5, one can characterize the conditions under which (26) degenerates to (1) via the behavior of first- and second-order derivatives of ξ_s in the neighborhood of $s = 0$.

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[11] See also footnote 5 and our remark after Theorem 5.

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出國報告（出國類別：出席會議）

出席 Psychonomic Society 年會及 Society for Judgment and Decision Making 年會

服務機關：國立台灣大學心理系
姓名職稱：徐永豐 副教授
派赴國家：Chicago, USA
出國期間：November 19–23, 2015
報告日期：December 6, 2015
經費來源：科技部

目的

The main purpose of this trip was to present our work in the 56th Annual Meeting of the Psychonomic Society, which began the evening of November 19th and ended the early afternoon of November 22nd in Chicago. During my stay in Chicago, I also took the opportunity to attend several talks in one of the affiliate meetings---the 36th annual meeting of the Society for Judgment and Decision Making (SJDM), held November 20–23 in the same conference hotel as the Psychonomic meeting but occupied a much smaller capacity. After the two conferences I flew to Southern California for a research stay for several days.

過程

In the Psychonomic conference, my student and I gave a poster presentation titled "Effects of Affective Arousal on Prediction Errors in Reinforcement Learning: Evidence from Feedback-Related Negativity". Here is the abstract.

"Equality and fairness in social interaction often elicit affective arousal and show a great impact on decision making. The present study aims to uncover the mechanism behind such daily-life experiences using the behavioral, model-fitting, and electrophysiological approaches. In the first session of the experiment, subjects were randomly assigned to one of the "Neutral," "Harsh," and "Kind" groups to undertake a different level of conceived fairness. Then a probabilistic learning task with varied probabilities of negative-reward assignments was applied to each subject to examine the impact of emotional experience on her/his choice behavior. Trial-by-trial data were fitted with a reinforcement learning (RL) model using the Bayesian approach. Brain activities were analyzed via ERPs. Our analyses revealed that, compared with other two groups, subjects in the "Harsh" group retained more task scores, exhibited distinct parameter values of the RL model, and demonstrated a higher degree of feedback-related negativity, suggesting a tendency to loss aversion."

Quite a number of people stopped by our poster and asked various questions. Some of the feedbacks are invaluable and will definitely help us improve the presentation of the research results. We are currently wrapping up the work and plan to submit the results for publication in the near future.

心得

Attending both the Psychonomic and SJDM conferences helps me keep track of the on-going research topics in the respective fields. For example, the various symposia held in the Psychonomic conference provided a glimpse of what people are caring about nowadays.

In the SJDM conference I attended a two-hour tribute to Paul Slovic, an event that attracted a big audience. Paul Slovic is a big figure in the field of judgment and decision making. He has made major contributions to research on preference construction, risk perception, and decision analysis. This 2-hour program highlighted Paul Slovic's influence across the fields. Several prominent speakers, including Daniel Kahneman, Baruch Fischhoff and others, gave short talks to pay tribute to him. Paul Slovic was there and also gave a commentary at the end.

I also attended several other SJDM talks on topics of collaboration and cooperation, evaluations of experiences, etc. I noticed that more and more scholars are using the Amazon Mechanical Turk for their online surveys and experiments. Also, the use of Bayesian analysis is getting widespread. However, while the use of neuroimaging techniques is getting popular for helping uncover the mechanism of choice behavior (i.e., neuroeconomics), there were relatively few talks or posters in the SJDM conference on combining neuroimaging with behavioral measures to study issues in judgment and decision making.

出國報告（出國類別：出席會議）

出席 47th European Mathematical Psychology Group 年會

服務機關：國立台灣大學心理系
姓名職稱：徐永豐 副教授
派赴國家：Copenhagen, Denmark
出國期間：June 18–22, 2016
報告日期：July 5, 2016
經費來源：科技部

目的

The purpose of this trip was to present my work in the 47th Annual Meeting of the European Mathematical Psychology Group (EMPG), which began the morning of June 20th and ended the afternoon of June 22nd in Copenhagen, Denmark. During my stay, I also attended two full-day EMPG-sponsored preconference workshops, held on June 18th & 19th in the same place. After the conference I flew to Paris, France for several days before heading back to Taiwan.

過程

As mentioned above, I attended two full-day EMPG-sponsored preconference workshops. The theme of the first workshop, solely lectured by Professor Jürgen Heller on June 18th, was on the introduction to the representational theory of measurement. The theme of the second workshop, jointly lectured by Professors Michel Regenwetter, Jean-Paul Doignon, Clintin Davis-Stober, and Daniel Cavagnaro on June 19th, was on the probabilistic specification and quantitative testing of decision theories.

In the EMPG meeting I gave an oral presentation titled "*Conditions under which affine representations are also Fechnerian under the power law of similarity*". Here is the abstract.

"In a recent paper, Hsu, Iverson, and Doble (2010) examined some properties of a (weakly balanced) affine representation for choices, $\Psi(x, y) = F[(u(x)-u(y))/\sigma(y)]$, and showed that using the Fechner method of integrating just-noticeable differences (jnds), one can reconstruct the scales u and σ from the behavior of (Weber) sensitivities $\xi_s(x) = x + \Delta_s(x)$ (where $s = F^{-1}(\Psi)$ and Δ_s is the jnd) in a neighborhood of $s = 0$. Following Iverson (2006), in this study we impose a power law of similarity on the sensitivities, $\xi_s(\lambda x) = \lambda^{\iota(s)}\xi_{\eta(\lambda,s)}(x)$, and study its impact on u and σ in the affine representation. Especially, we specify the conditions for the first- and second-order derivatives of $\xi_s(x)$ with respect to s (and evaluated at $s \to 0$) under which the affine representation degenerates to a Fechnerian one. We also link the results to the solutions in Iverson (2006) that was worked out within the Fechnerian framework."

There were about 40 people attending my talk. The feedback from several participants was informative; it gave me some insights into how the research could be proceeded/expanded in the future.

心得

Attending the two workshops is worthwhile. In the past I had learned some of the topics covered in the workshops and also had attended some of the related talks in various occasions, but this time the two workshop themes were introduced in a very coherent and systematic way. I truly enjoyed and also learned a lot from both workshops.

Another thought is that the EMPG meeting has smaller size in attendance than the (similar) US-based annual meeting of the Society for Mathematical Psychology; nonetheless, the EMPG meeting is more focused and provides more opportunities to interact intellectually with other scholars. For example, I got a chance to talk to Professor Jürgen Heller about measurement issues and also about possible future collaboration. I also met Professor Miguel Ángel García-Pérez and discussed possible optimal response formats for threshold estimation in psychophysics. Some of the conference talks were inspiring too. For example, Professor Matthias Gondan gave a convincing argument about why we shouldn't simply discard incorrect response times, and provided some interesting imputation methods for remedy. Overall, attending the EMPG meeting turned out to be a very rewarding experience for me.

科技部補助計畫衍生研發成果推廣資料表

日期:2015/12/06

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104年度專題研究計畫成果彙整表

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<tr><td colspan="2">研討會論文</td><td>1</td><td>Hsu, Y.-F., & Iverson, G. J. (2016). Conditions under which affine representations are also Fechnerian under the power law of similarity. Paper presented at the 2016 meeting of the European Mathematical Psychology Group, Copenhagen, Denmark.</td></tr>
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科技部補助專題研究計畫成果自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現（簡要敘述成果是否具有政策應用參考價值及具影響公共利益之重大發現）或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估
   ■達成目標
   □未達成目標（請說明，以100字為限）
   　□實驗失敗
   　□因故實驗中斷
   　□其他原因
   說明：

2. 研究成果在學術期刊發表或申請專利等情形（請於其他欄註明專利及技轉之證號、合約、申請及洽談等詳細資訊）
   論文：■已發表　□未發表之文稿　□撰寫中　□無
   專利：□已獲得　□申請中　■無
   技轉：□已技轉　□洽談中　■無
   其他：（以200字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性，以500字為限）
   The scientific merit of this research is that it contributes to the understanding of psychophysical representations (or scales) related to the Fechnerian framework -- a long-standing interest that follows closely the tradition of mathematical psychology.

4. 主要發現
   本研究具有政策應用參考價值：■否　□是，建議提供機關
   （勾選「是」者，請列舉建議可提供施政參考之業務主管機關）
   本研究具影響公共利益之重大發現：■否　□是
   說明：（以150字為限）
   We specify the conditions of the power law of similarity under which the weakly balanced affine representation degenerates to a Fechnerian one.